# ECE 901
# Homework 5
# The Curse of Dimensionality and the Blessing of Smoothness

Extend the denoising problem consider in **Lecture 15 — Denoising in Smooth Functions Spaces II: Adapting to Unknown Smoothness** to handle denoising of $d$ dimensional functions belonging to Hölder-$\alpha$, $\alpha > 0$, spaces. Show that maximum complexity regularized likelihood estimators (based on piecewise degree-$\lfloor\alpha\rfloor$ polynomial fits) converge at a rate of

$$O\left(n^{-\frac{2\alpha}{2\alpha+d}}\right)$$

Notice that in high dimensions, the rate can be very slow. This is the so-called *Curse of Dimensionality*. On the otherhand, we can re-arrange the exponent and write the rate as

$$O\left(n^{-\frac{2}{2+d/\alpha}}\right)$$

From this expression, it is easy to see that if $d' \equiv d/\alpha$ is small, then the rate approaches the parametric rate of $n^{-1}$. We can think of $d'$ as the "effective dimensionality" of the problem. If the unknown function is very smooth (large $\alpha$) this can offset the curse of dimensionality. This effect is sometimes called the *Blessing of Smoothness*.

**Hint:** Hölder-$\alpha$ smooth functions are functions whose $\lceil\alpha - 1\rceil$ derivative is Hölder-$(\alpha - \lfloor\alpha\rfloor)$. Therefore, for $\alpha \leq 1$ we used piecewise constant estimators and for $1 < \alpha \leq 2$ we used piecewise linear estimators. If $n < \alpha \leq n + 1$, then consider piecewise degree-$n$ polynomial estimators. Show that the squared bias of an $m$-piece degree-$n$ polynomial approximation is $O(m^{-2\alpha/d})$. From here it is easy to derive the overall rate for the MSE.