ECE 830 Fall 2010 Statistical Signal Processing

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Lecture 8: Signal Detection and Noise Assumption

1 Signal Detection

$$\begin{aligned} H_0: X &= W \\ H_1: X &= S + W \end{aligned}$$

where $W \sim N(0, \sigma^2 I_{n\times n})$ and $S = [s_1, s_2, \ldots, s_n]^T$ is the known signal waveform.

$$\begin{aligned} P_0(X) &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp(-\frac{1}{2\sigma^2} X^T X) \\ P_1(X) &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} exp[-\frac{1}{2\sigma^2}(X-S)^T(X-S)] \end{aligned}$$

The second equation holds because under hypothesis H_0, $W = X - S$.

The Log Likelihood Ratio test is

$$\log \Lambda(x) = \log \frac{P_W(X)}{P_W(X-S)} = -\frac{1}{2\sigma^2}[(X-S)^T(X-S) - X^T X] = -\frac{1}{2\sigma^2}[-2X^T + S^T S] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

After simplifying it, we can get

$$X^T S \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \gamma' + \frac{S^T S}{2} = \gamma$$

In this case, $X^T S$ is the sufficient statistics $t(x)$ for the parameter $\theta = 0, 1$. Note that $S^T S = \|S\|_2^2$ is the signal energy. The LR detector "filters" data by projecting them onto signal subspace.

1.1 Example 1

Suppose we want to control the probability of false alarm. For example, choose γ so that $\mathbb{P}(X^T S > \gamma \mid H_0) \leq 0.05$.

The test statistic $X^T S$ is usually called "matched filter".

In particular, projection onto subspace spanned by S is

$$P_S = \frac{SS^T}{S^T S} = \frac{S}{\|S\|} \cdot \frac{S^T}{\|S\|}$$

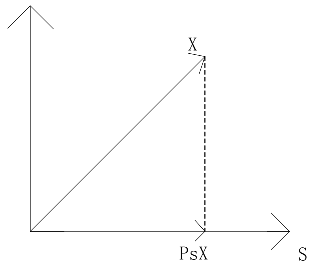

Figure 1: Projection of X onto subspace S

$$P_S X = \frac{SS^T}{\|S\|^2} X = (X^T S) \frac{S}{\|S\|^2}$$

where $\frac{X^T S}{\|S\|^2}$ is just a number.

Geometrically, suppose the horizontal line is the subspace S and X is some other vector. The projection of vector X into subspace S can be expressed in the figure 1.

1.2 Example 2

Suppose the signal value S_k is sinusoid.

$$S_k = \cos(2\pi f_0 k + \theta), k = 1, \ldots, n$$

The match filter in this case is to compute the value in the specific frequency. So P_S in this example is a bandpass filter.

1.3 Performance Analysis

Next problem what we want to know is what's the probability density of $X^T S$, which is the sufficient statistics of this test.

$$\begin{aligned} H_0 : X &\sim N(0, \sigma^2 I) \\ H_1 : X &\sim N(S, \sigma^2 I) \end{aligned}$$

$X^T S = \sum_{k=1}^{n} X_k S_k$ is also Gaussian distributed. Recall if $X \sim N(\mu, \Sigma)$, then $Y = AX \sim N(A\mu, A\Sigma A^T)$, where A is a matrix.

Since $Y = X^T S = S^T X$, Y is a scalar. So we can get

$$\begin{aligned} H_0 : X^T S &\sim N(0^T S, S^T \sigma^2 I S) = N(0, \sigma^2 \|S\|^2) \\ H_1 : X^T S &\sim N(S^T S, S^T \sigma^2 I S) = N(\|S\|^2, \sigma^2 \|S\|^2) \end{aligned}$$

The probability of false alarm is $P_{FA} = Q(\frac{\gamma - 0}{\sigma \|S\|})$, and the probability of detection is $P_D = Q(\frac{\gamma - \|S\|^2}{\sigma \|S\|}) = Q(\frac{\gamma}{\sigma \|S\|} - \frac{\|S\|}{\sigma})$. Since Q function is invertible, we can get $\frac{\gamma}{\sigma \|S\|} = Q^{-1}(P_{FA})$. Therefore, $P_D = Q(Q^{-1}(P_{FA}) - \frac{\|S\|}{\sigma})$. In the equation, $\frac{\|S\|}{\sigma}$ is the square root of Signal Noise Ratio($\sqrt{SNR}$).

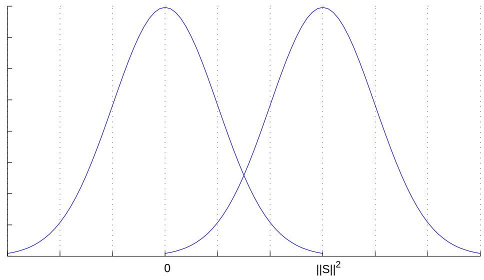

Figure 2: Distribution of P_0 and P_1

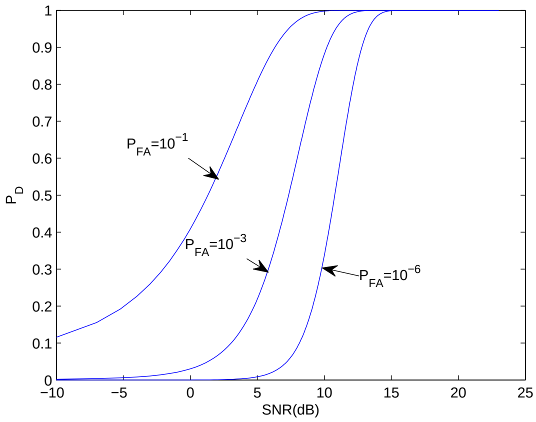

Figure 3: Relation between probability of detection and false alarm

2 AWGN Assumption

Is real-world noise really additive, white and Gaussian? Well, here are a few observations. Noise in many applications (e.g. communication and radar) arose from several independent sources, all adding together at sensors and combining additively to the measurement. AWGN is gaussian distributed as the following formula.

$$W \sim N(0, \sigma^2 I)$$

CLT(Central Limit Theorem): If $x_1, \dots, x_n$ are independent random variables with means μ_i and variances $\sigma_i^2 < \infty$,then $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i} \to N(0,1)$ in distribution quite quickly.

Thus, it is quite reasonable to model noise as additive and Gaussian list in many applications. However, whiteness is not always a good assumption.

2.1 Example 3

Suppose $W = S_1 + S_2 + \cdots + S_k$, where $S_1, S_2, \dots S_k$ are inteferencing signals that are not of interest. But each of them is structured/correlated in time. Therefore, we need a more generalized form of noise, which is "Colored Gaussian Noise".

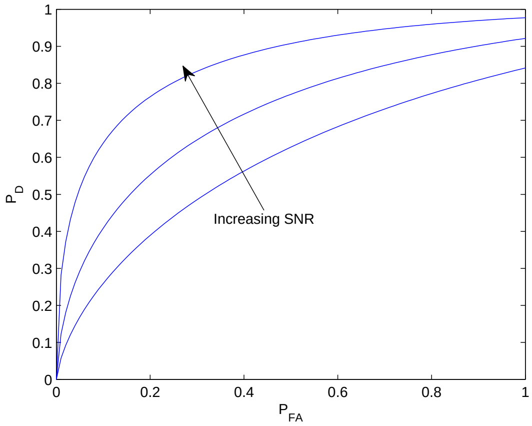

Figure 4: Relation between probability of detection and SNR

3 Colored Gaussian Noise

$W \sim N(0, \Sigma)$ is called correlated or "colored" noise, where Σ is a structured covariance matrix.

Consider the binary hypothesis test in this case.

$$\begin{aligned} H_0: \ X &= S_0 + W \\ H_1: \ X &= S_1 + W \end{aligned}$$

where $W \sim N(0, \Sigma)$ and S_0 and S_1 are know signal waveforms. So we can rewrite the hypothesis as

$$\begin{aligned} H_0: \ X \sim N(S_0, \Sigma) \\ H_1: \ X \sim N(S_1, \Sigma) \end{aligned}$$

The probability density of each hypothesis is

$$P_i(X) = \frac{1}{(2\pi)^{\frac{2}{n}}(\Sigma)^{\frac{1}{2}}} exp[-\frac{1}{2}(X - S_i)^T \Sigma^{-1}(X - S_i)], i = 0, 1$$

The log likelihood ratio is

$$\log(\frac{P_1(X)}{P_2(X)}) = -\frac{1}{2}[(X-S_1)^T \Sigma (X-S_1) - (X-S_0)^T \Sigma^{-1}(X-S_0)] = X^T \Sigma^{-1}(S_1-S_0) - \frac{1}{2} S_1^T \Sigma^{-1} S_1 + \frac{1}{2} S_0^T \Sigma^{-1} S_0 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$(S_1 - S_0)\Sigma^{-1} X \underset{H_0}{\overset{H_1}{\gtrless}} \gamma' + \frac{S_1^T \Sigma^{-1} S_1}{2} - \frac{S_0^T \Sigma^{-1} S_0}{2} = \gamma$$

Let $t(X) = (S_1 - S_0)\Sigma^{-1} X$, we can get

$$\begin{aligned} H_0: \ t \sim N((S_1 - S_0)\Sigma^{-1} S_0, (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)) \\ H_1: \ t \sim N((S_1 - S_0)\Sigma^{-1} S_1, (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)) \end{aligned}$$

The probability of false alarm is

$$P_{FA} = Q(\frac{\gamma - (S_1 - S_0)^T \Sigma^{-1} S_0}{[(S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)]^{\frac{1}{2}}})$$

In this case it is natural to define

$$SNR = (S_1 - S_0)^T \Sigma^{-1}(S_1 - S_0)$$

3.1 Example 4

$S_1 = [\frac{1}{2}, \frac{1}{2}]$, $S_0 = [-\frac{1}{2}, -\frac{1}{2}]$, $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$.

The test statistics is

$$y = (S_1 - S_0)\Sigma^{-1} X = [1, 1]\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{1+\rho}(x_1 + x_2)$$

$$\begin{aligned} H_0 : \ & y \sim N(-\frac{1}{1+\rho}, \frac{2}{1+\rho}) \\ H_1 : \ & y \sim N(+\frac{1}{1+\rho}, \frac{2}{1+\rho}) \end{aligned}$$

The probability of false alarm is

$$P_{FA} = Q(\frac{\gamma + \frac{1}{1+\rho}}{\sqrt{\frac{2}{1+\rho}}})$$

The probability of detection is

$$P_D = Q(\frac{\gamma - \frac{1}{1+\rho}}{\sqrt{\frac{2}{1+\rho}}})$$

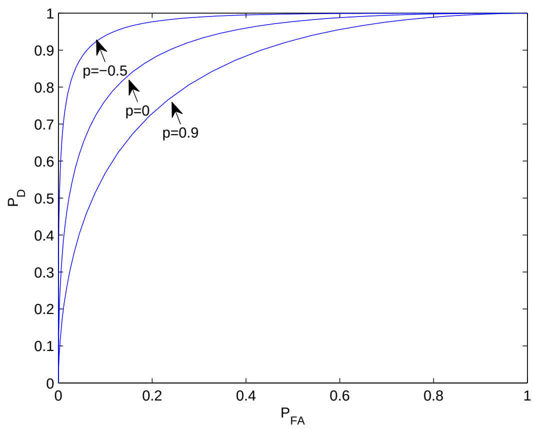

Figure 5: ROC curve at different ρ