ECE 830 Fall 2010 Statistical Signal Processing

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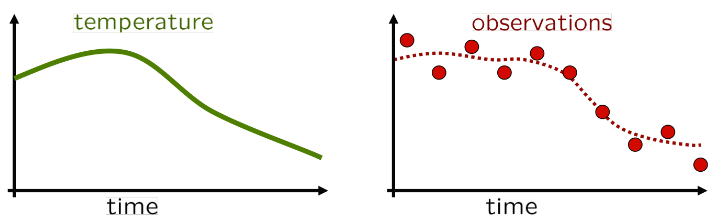
Lecture 11: Dynamic Filtering

1 Dynamic Filtering

Motivation: In many applications we want to track a time-varying (dynamic) phenomenon.

Example 1 *Tracking temperature or humidity in a museum room with an inaccurate device.*



Key: Temperature changes slowly with time so we should be able to average across time to obtain better estimates. How to do this? Model dynamics of temperature changes and noise/uncertainties in measurement.

2 Dynamical State Equation (Prior)

$x_1, x_2, \ldots$ denotes quantity ("state") of interest.

Assume,

$$p(x_n|x_{n-1}, \ldots, x_1) = p(x_n|x_{n-1})$$

Where p($x_n|x_{n-1}$) is the state transition probability function. We make the Markovian assumption that the distribution of x_n depends only on x_{n-1}.

To define the state process we need to specify
(a) p(x_1) (initial "state" distribution)
(b) p($x_n|x_{n-1}$) n = 2,3,... (state transition probability density functions)

Example 2 *Santa Tracker*

x(t) = santa's position at time t on Christmas Eve
$\frac{\partial x(t)}{\partial t} = v(t)$ *velocity, Brownian motion*
$\frac{\partial v(t)}{\partial t} = u(t)$ *Acceleration, GWN process*

$$\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma^2 \end{bmatrix} u_n, \; u_n \sim \mathcal{N}(0,1), \Delta \textit{ small}$$

3 Observation Model (Likelihood)

Usually we cannot observe x_n directly. Instead we observe z_1,z_2, ...

$$z_n = x_n + w_n, w_n \sim \mathcal{N}(0, \sigma^2)$$

$$z_n \;=\; A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} + w_n, \text{A is a 2x2 matrix}$$

$$z_n = f(x_n) + w_n \text{ f is a non-linear function}$$

Let p($z_n|x_n$) denote the likelihood of x_n based on observation z_n.

Posterior:

$$p(x|z) \propto p(z|x)p(x)$$

Key: Posterior can be computed efficiently in an incremental fasion by exploiting Markovian structure of state transitions (prior).

4 Density Propagation

An incremental procedure for efficiently computing p($x_n|z_1$, ...,z_n).

Notation:

Prior:

$$S_n(x_n|x_{n-1}) := p(x_n|x_{n-1}), \quad P_1(x_1) = p(x_1)$$

Likelihood:

$$L_n(z_n|x_n) := p(z_n|x_n)$$

Posterior:

$$F_n(x_n) := p(x_n|z_1, \ldots, z_n)$$

Prediction:

$$P_n(x_n) := p(x_n|z_1, \ldots, z_{n-1})$$

5 Density Propogation Algorithm

n = 1:
Predict x_1:

$$x_1 \sim p_1(x_1)$$

Observe z_1
Compute Posterior:

$$F_1(x_1) = p(x_1|z_1) = \frac{p(z_1|p(x_1)}{p(z_1)} \propto L_1(z_1|x_1)p_1(x_1)$$

n = 2:
Predict x_2:

$$\begin{aligned}
p(x_1, x_2|z_1) &= \frac{p(x_1, x_2, z_1)}{p(z_1)} \\
&= \frac{p(x_2|x_1, z_1)p(x_1|z_1)p(z_1)}{p(z_1)} \\
&= p(x_2|x_1)F_1(x_1) \\
&= S_2(x_2|x_1)F_1(x_1) \\
p(x_2|z_1) &= \int S_2(x_2|x_1)F_1(x_1)\mathrm{d}x \\
&=: P_2(x_2)
\end{aligned}$$

Observe z_2
Compute Posterior:

$$\begin{aligned}
F_2(x_2) &= p(x_2|z_1, z_2) \\
&= \frac{p(x_1, z_1, z_2)}{p(z_1, z_2)} \\
&= \frac{p(z_2|x_2)p(x_2|z_1)p(z_1)}{p(z_1, z_2)} \\
&\propto L_2(z_2|x_2)P_2(x_2)
\end{aligned}$$

In general, at time step n:
Predict x_n:

$$\begin{aligned}
P_n(x_n) &= p(x_n|z_1, \ldots, z_{n-1}) \\
&= \int S_n(x_n|x_{n-1})F_{n-1}(x_{n-1})\mathrm{d}x_{n-1}
\end{aligned}$$

Observe z_n
Compute Posterior:

$$\begin{aligned}
F_n(x_n) &= p(x_n|z_1, \ldots, z_n) \\
&\propto L_n(z_n|x_n)P_n(x_n)
\end{aligned}$$

5.1 Block Diagram

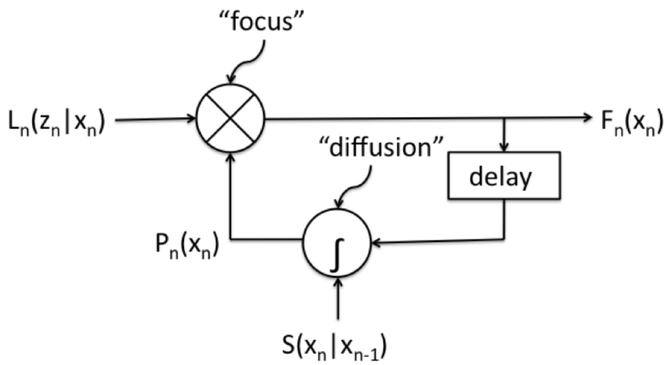


Figure 1: Block diagram of dynamic filtering.

5.2 Filtering

$$F_n(x_n) = L_n(z_n|x_n)P_n(x_n)$$

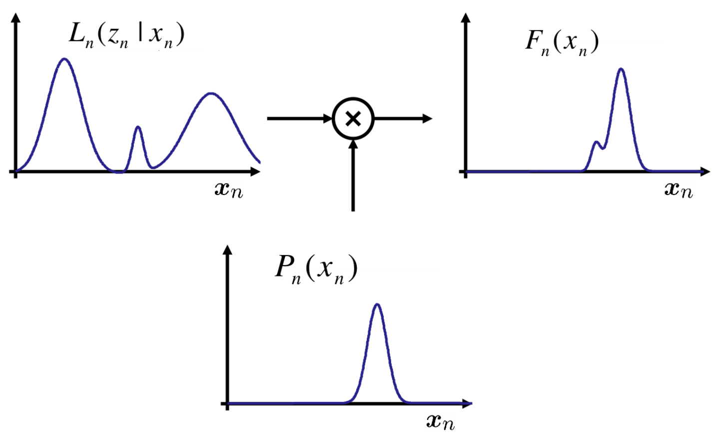


Figure 2: The filtering or "focus" portion of the dynamical filtering block diagram.

5.3 Prediction

$$P_{n+1}(x_{n+1}) = \int S_n(x_{n+1}|x_n)F_n(x_n)\mathrm{d}x_n$$

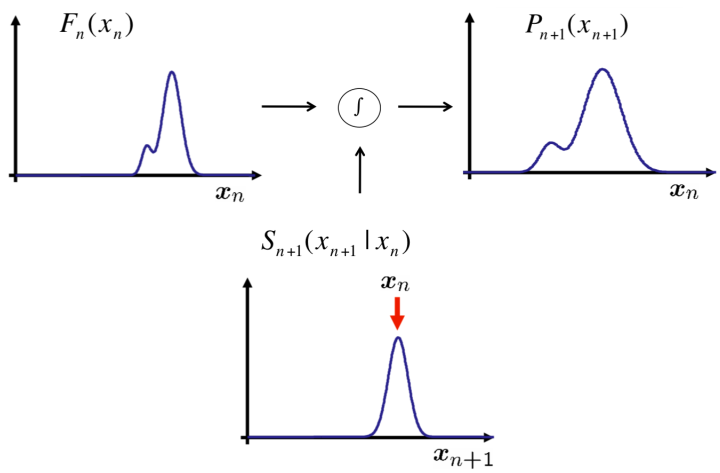


Figure 3: The "diffusion" portion of the dynamical filtering block diagram.

6 Estimating x_n

We have many possibilities. Given,

$$F_n(x_n) = p(x_n|z_1, \ldots, z_n)$$

We can minimize various loss functions.

ℓ_2:

$$\begin{aligned}\hat{x}_n &= \mathbb{E}_{F_n}[(x_n - \hat{x}_n)^2] \\ &= \int x_n F_n(x_n)\mathrm{d}x_n\end{aligned}$$

ℓ_1:

$$\hat{x}_n = \mathbb{E}_{F_n}[|x_n - \hat{x}_n|]$$

$\ell_{0/1}$:

$$\hat{x}_n = \arg\max_x F_n(x_n)$$