Performance of the Periodogram

Bias Analysis

$$E\left[\hat{\Gamma}_p(f)\right] = E\left[\hat{\Gamma}_c(f)\right]$$

$$= \sum_{k=-(N-1)}^{N-1} E\left[\hat{\gamma}(k)\right] e^{-j2\pi f k}$$

For $0 \le k \le N-1$,

$$E\left[\hat{\gamma}(k)\right] = \frac{1}{N} E\left[\sum_{n=k+1}^{N} x(n)\, x^*(n-k)\right].$$

$$= \frac{1}{N}(N-k)\,\gamma(k) = \left(1 - \frac{k}{N}\right)\gamma(k)$$

For $-(N-1) \le k < 0$,

$$\hat{\gamma}(k) \equiv \hat{\gamma}^*(-k)$$ (proper symmetry of covariance function)

and

$$E\left[\hat{\gamma}(k)\right] = \left(1 - \frac{|k|}{N}\right)\gamma^*(-k)$$

$$= \left(1 - \frac{|k|}{N}\right)\gamma(K)$$

This gives

$$E\left[\hat{\Gamma}_p(f)\right] = \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \gamma(k)\, e^{-j2\pi f k}$$

Now define

$$w_B(k) = \begin{cases} 1 - \frac{|k|}{N} & , \quad k = 0, \pm 1, \ldots, \pm(N-1) \\ 0 & , \quad \text{otherwise} \end{cases}$$

We see that

$$E\left[\hat{\Gamma}_p(f)\right] = \sum_{k=-\infty}^{\infty} w_B(k)\, \gamma(k)\, e^{-j2\pi f k}$$

What is the Fourier transform of the sequence $\{ w_B(k)\, \gamma(k) \}$?

$$E\left[\hat{\Gamma}_p(f)\right] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma(\nu)\, W_B(f-\nu)\, d\nu$$

Some simple calculations show that

$$W_B(f) = \frac{1}{N}\left[\frac{\sin(\pi f N)}{\sin(\pi f)}\right]^2$$

$\{w_B(k)\}$ is called the triangular window or <u>Bartlett Window</u>.

$W_B(f)$ is called the <u>Fejer kernel</u>.

<u>Remark</u>: The expected value of the periodogram is the true spectral density convolved with the Fejer kernel.

The bias of the periodogram is given by

$$\text{bias}\left(\hat{\Gamma}_P(f)\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma(\nu) W_B(f-\nu)\, d\nu - \Gamma(f)$$

Clearly, if $W_B(f)$ were an ideal Dirac impulse $\delta(f)$, then the bias would be exactly zero.

However, simple calculations show that the main center lobe of $W_B(f)$ has a width of approximately $\frac{1}{N}$ Hz.

Exercise:

Find the "half-power" width of the main lobe of a window function $W(f)$.

The half-power cut-off frequency is the value of f satisfying

$$W(f) = \frac{W(0)}{2}$$

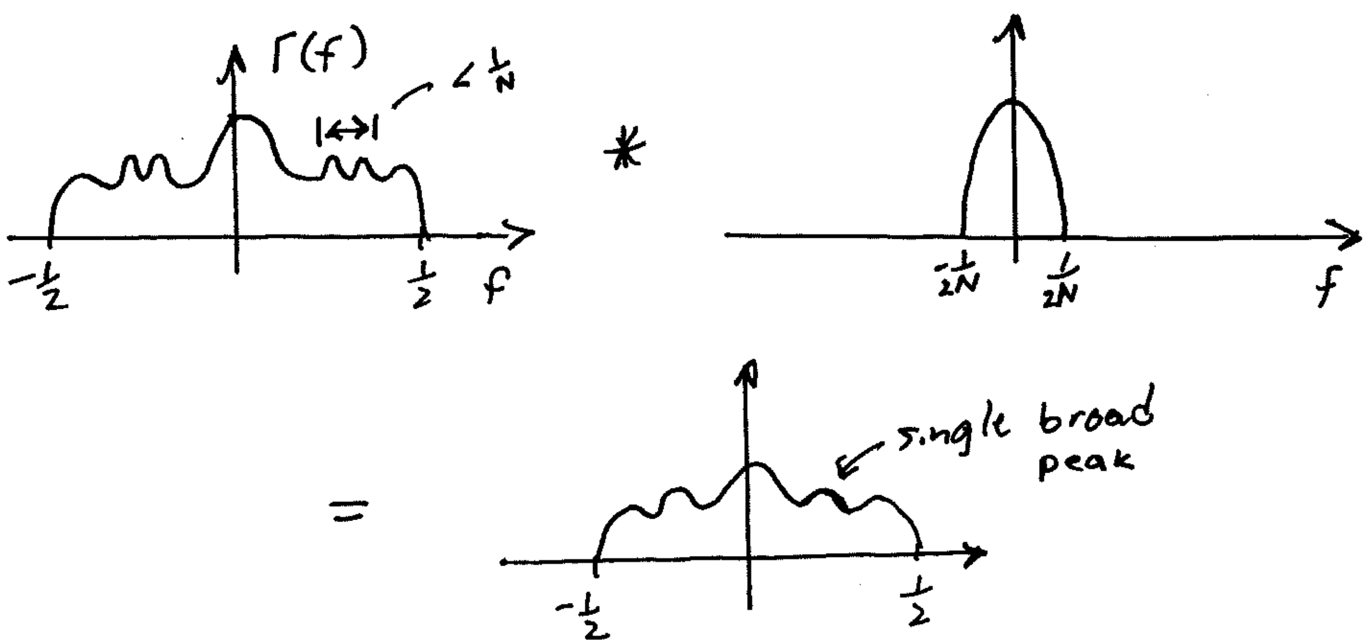
Note that a general window function has a peak at $f=0$ and is symmetric about that point. Furthermore, assume that the peak of $W(f)$ is sufficiently narrow (as it should be).

Then using a Taylor series expansion, show that the bandwidth (half-power) of the main lobe is approximately

$$B \approx 2\sqrt{\frac{W(0)}{|W''(0)|}}$$

Effect of Main Lobe

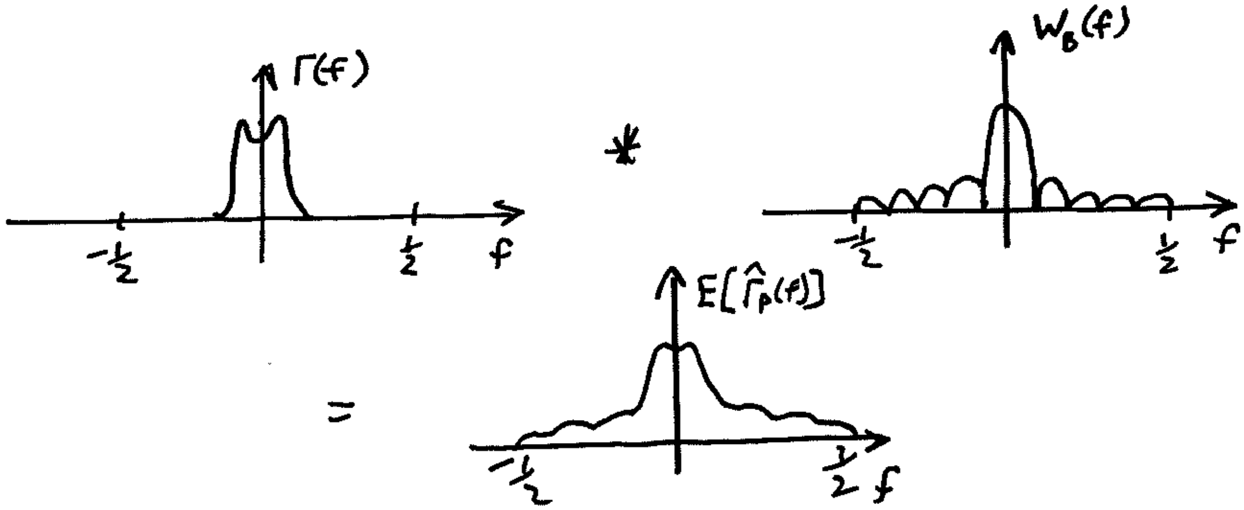
The main lobe of $W_B(f)$ smears or smoothes the mean of the periodogram. For example, if $\Gamma(f)$ has two peaks separated by less than $\frac{1}{N}$ Hz, then these two peaks will appear as a single broader peak in the periodogram.



Because of this smearing effect, periodogram estimators cannot resolve details in the spectrum that are separated by less than $\frac{1}{N}$ Hz. For this reason, $\frac{1}{N}$ is called the <u>spectral resolution</u> limit of the periodogram.

Effect of Sidelobes

The sidelobes cause a transition of power from the spectral bands with the most power to bands that contain less or no power.



Asymptotic Bias – Large N

$$\lim_{N \to \infty} E\left[\hat{\Gamma}_p(f)\right] = \Gamma(f)$$

" the periodogram is an <u>asymptotically unbiased estimator</u> "

<u>Check</u>: consider

$$E\left[\hat{\Gamma}_p(f)\right] = \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) r(k)\, e^{-j2\pi f k}$$

So, if bias were are only concern, then simply increasing N would eliminate our problems.

Unfortunately, the real problem of the periodogram is its large variance.

Variance Analysis

The finite-sample variance of $\hat{\Gamma}_p(f)$ is difficult to analyze except in special cases (e.g. white noise).

Therefore, we will focus on the asymptotic variance (as $N \to \infty$).

This will demonstrate the poor statistical accuracy of $\hat{\Gamma}_p(f)$, even when a large number of samples are available.

Preliminaries

A sequence $\{e(n)\}$ is called <u>complex</u> (or <u>circular</u>) <u>white noise</u> if it satisfies

(i) $E[e(n)e^*(m)] = \sigma^2 \delta(n-m)$

(ii) $E[e(n)e(m)] = 0$ for all n and m

Note that this implies that

$\{Re(e(n))\}$ and $\{Im(e(n))\}$

are uncorrelated real white noise processes, each having power $\sigma^2/2$.

First let's consider the case where the process under study is a Gaussian complex white noise.

Recall that for white noise $\Gamma(f) = \sigma^2$.

We will establish the following result.

$$\text{(☆)} \quad \lim_{N\to\infty} E\left[\left(\hat{\Gamma}_p(f_1) - \Gamma(f_1)\right)\left(\hat{\Gamma}_p(f_2) - \Gamma(f_2)\right)\right] = \begin{cases} \Gamma^2(f_1), & f_1 = f_2 \\ 0, & f_1 \neq f_2 \end{cases}$$

Since we have already verified that

$$\lim_{N\to\infty} E\left[\hat{\Gamma}_p(f)\right] = \Gamma(f)$$

to prove (☆) it suffices to show that

$$\text{(\#)} \quad \lim_{N\to\infty} E\left[\hat{\Gamma}_p(f_1)\hat{\Gamma}_p(f_2)\right] = \Gamma(f_1)\Gamma(f_2) + \Gamma^2(f_1)\delta(f_1 - f_2)$$

Now

$$\hat{\Gamma}_p(f) = \frac{1}{N}\left|\sum_{n=1}^{N} e(n)\, e^{-j2\pi f n}\right|^2$$

So

$$E\left[\hat{\Gamma}_p(f_1)\,\hat{\Gamma}_p(f_2)\right] = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N} E\left[e(k)e^*(l)e(m)e^*(n)\right] \times e^{-j2\pi f_1(k-l)}\, e^{-j2\pi f_2(m-n)}$$

<u>Fact 1</u>: (see Papoulis (1984))

If w, x, y, z are jointly Gaussian (real or complex) random variables, then

$$E[wxyz] = E[wx]\cdot E[yz] + E[wy]\cdot E[xz] + E[wz]\cdot E[xy] + E[w]\cdot E[x]\cdot E[y]\cdot E[z]$$

Therefore, since $\{e(k)\}$ is a complex Gaussian white noise:

$$\begin{aligned} E\left[e(k)e^*(l)e(m)e^*(n)\right] &= E\left[e(k)e^*(l)\right]\cdot E\left[e(m)e^*(n)\right] \\ &+ E\left[e(k)e(m)\right]\cdot E\left[e^*(l)e^*(n)\right] \\ &+ E\left[e(k)e^*(n)\right]\cdot E\left[e^*(l)e(m)\right] \\ &= \sigma^4\left(\delta(k-l)\cdot\delta(m-n) + \delta(k-n)\cdot\delta(l-m)\right) \end{aligned}$$

This yields

$$E\left[\hat{\Gamma}_p(f_1)\,\hat{\Gamma}_p(f_2)\right] = \sigma^4 + \frac{\sigma^4}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N} e^{-j2\pi(f_1-f_2)(k-l)}$$

$$= \sigma^4 + \frac{\sigma^4}{N^2}\left|\sum_{k=1}^{N} e^{-j2\pi(f_1-f_2)k}\right|^2$$

$$= \sigma^4 + \underbrace{\frac{\sigma^4}{N^2}\left(\frac{\sin((f_1-f_2)\pi N)}{\sin(\pi(f_1-f_2))}\right)^2}$$

$$\longrightarrow \sigma^4\,\delta(f_1-f_2) \quad \text{as } N \to \infty$$

Thus, for white noise,

$$\lim_{N\to\infty} E\left[\hat{\Gamma}_p(f_1)\,\hat{\Gamma}_p(f_2)\right] = \begin{cases} \Gamma^2(f_1) = 2\sigma^4, & f_1 = f_2 \\ \sigma^4, & f_1 \neq f_2 \end{cases}$$

$\Rightarrow$ the variance of $\hat{\Gamma}_p(f) = \sigma^4$, a <u>constant</u>!

$\Rightarrow$ variance does not decrease as $N \to \infty$, and therefore $\hat{\Gamma}_p(f)$ is <u>inconsistent</u>.

Now, for the final step in our variance analysis, consider a more general complex Gaussian process obtained by passing a Gaussian white noise through a linear, time-invariant filter:

$$x(n) = \sum_{k=1}^{\infty} h(k)\, e(n-k)$$

The power spectral density of x is

$$\Gamma_{xx}(f) =$$

where

$$H(f) =$$

We will show that

$$\hat{\Gamma}_{xx}(f) = |H(f)|^2 \hat{\Gamma}_{ee}(f) + \eta(f) \quad (**)$$

where $\eta(f)$ is a random variable satisfying

$$E[\eta^2(f)] \le \frac{\text{Constant}}{N}$$

Thus,

$$\lim_{N\to\infty} \hat{\Gamma}_{xx}(f) = |H(f)|^2 \left(\lim_{N\to\infty} \hat{\Gamma}_{ee}(f) \right)$$

and

$$\lim_{N\to\infty} E\left[\left(\hat{\Gamma}_{xx}(f_1) - \Gamma_{xx}(f_1)\right)\left(\hat{\Gamma}_{xx}(f_2) - \Gamma(f_2)\right)\right]$$

$$= \begin{cases} \Gamma^2(f_1), & f_1 = f_2 \\ 0, & f_1 \ne f_2 \end{cases}$$

To prove 44, first note that

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x(n)\, e^{-j2\pi f n} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\sum_{k=1}^{\infty} h(k)\, e(n-k)\, e^{-j2\pi(n-k)} e^{-j2\pi k}$$

$$= \frac{1}{\sqrt{N}}\sum_{k=1}^{\infty} h(k)\, e^{-j2\pi f k} \sum_{p=1-k}^{N-k} e(p)\, e^{-j2\pi f p}$$

$$= \frac{1}{\sqrt{N}}\sum_{k=1}^{\infty} h(k)\, e^{-j2\pi f k} \times$$

$$\left[\sum_{p=1}^{N} e(p)\, e^{-j2\pi f p} + \underbrace{\sum_{p=1-k}^{0} e(p)\, e^{-j2\pi f p} - \sum_{p=N-k+1}^{N} e(p)\, e^{-j2\pi f p}}_{\equiv\, g_k(f)} \right]$$

$$= H(f)\, \frac{1}{\sqrt{N}} \sum_{p=1}^{N} e(p)\, e^{-j2\pi f p} + \underbrace{\frac{1}{\sqrt{N}} H(f)\, g_k(f)}_{\equiv\, \rho(f)}$$

Note that

$$E[\,q_k(f)\,] = 0$$

$$E[\,q_k(f)\,q_\ell(f)\,] = 0 \quad \text{for all } k \text{ and } \ell$$

$$E[\,q_k(f)\,q_\ell^*(f)\,] = 2\sigma^2 \min(k,\ell)$$

$2\min(k,\ell)$ is the number of terms of the form $E[\,|e(p)|^2\,]$

Which imply

$$E[\,e(f)\,] = 0$$

$$E[\,e^2(f)\,] = 0$$

$$\left(\text{but} \quad E[\,|e(f)|^2\,] \neq 0\right)$$

$$E\left[|e(f)|^2\right]$$

$$= \frac{1}{N}\left| \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} h(k)e^{-j2\pi fk}\, h^*(l)e^{j2\pi fl} E\left[q_k(f)\, q_l^*(f)\right]\right|$$

$$= \frac{2\sigma^2}{N}\left| \sum_{k=1}^{\infty} h(k)e^{-j2\pi fk}\left(\sum_{l=1}^{k} h^*(l)\cdot l\cdot e^{j2\pi fl} + \sum_{l=k+1}^{\infty} h^*(l)\cdot k e^{-j2\pi fl}\right)\right|$$

$$\leq \frac{2\sigma^2}{N}\sum_{k=1}^{\infty} |h(k)| \left(\sum_{l=1}^{\infty} |h(l)|\cdot l + \sum_{l=1}^{\infty} |h(l)|\cdot k \right)$$

$$= \frac{4\sigma^2}{N}\left(\sum_{k=1}^{\infty}|h(k)|\right)\left(\sum_{l=1}^{\infty} l\cdot |h(l)|\right)$$

Now, provided $\sum_{l=1}^{\infty} l\cdot|h(l)|$ is finite

(which requires the impulse response to decay sufficiently fast),

we have

$$E\left[|e(f)|^2\right] \leq \frac{\text{Constant}}{N}$$

Now,

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} x(n)\, e^{-j2\pi f n} = H(f) \cdot \left[\frac{1}{\sqrt{N}} \sum_{p=1}^{N} e(p) e^{-j2\pi f p} \right] + \rho(f)$$

yields

$$\hat{\Gamma}_{xx}(f) = \left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} x(n) e^{-j2\pi f n} \right) \left(\frac{1}{\sqrt{N}} \sum_{m=1}^{N} x(m) e^{-j2\pi f m} \right)^{*}$$

$$= |H(f)|^2 \cdot \hat{\Gamma}_{ee}(f) + \eta(f)$$

where

$$\eta(f) = H^{*}(f) E^{*}(f) \rho(f) + H(f) E(f) \rho^{*}(f) + \rho(f) \rho^{*}(f)$$

and

$$E(f) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e(n)\, e^{-j2\pi f n}$$

Since $E(f)$ and $e(f)$ are linear combinations of Gaussian variables, they themselves are Gaussian distributed.

Hence, we can apply the fourth-order moment formula (Fact 1) to compute the second-order moment of $\eta(f)$. Using the formula and the fact that $E\left[|e(f)|^2\right] = \frac{\text{const.}}{N}$ and the Schwarz inequality

$$\left| E[e(f)] \cdot E^*[E(f)] \right|^2 \le E\left[|e(f)|^2\right] \cdot E\left[|E(f)|^2\right]$$

$$= \frac{\text{const.}}{N} \cdot \underbrace{E\left[\left|\hat{\Gamma}_{ee}(f)\right|^2\right]}_{\text{Const.}}$$

we can verify that

$$E\left[|\eta(f)|^2\right] \le \frac{\text{Const}}{N}$$

$$\Rightarrow \quad \eta(f) \to 0 \quad \text{as } N \to \infty$$

with prob. 1

Summary

If x is a complex, stationary, Gaussian process and if $\Gamma_{xx}(f)$ is sufficiently smooth

$$\left(\sum k\,|h(k)| < \infty \Rightarrow H(f),\ \Gamma_{xx}(f) \text{ are smooth} \right)$$

then

$$\lim_{N\to\infty} E\left[\left(\hat{\Gamma}_{xx}(f_1)-\Gamma_{xx}(f_1)\right)\left(\hat{\Gamma}_{xx}(f_2)-\Gamma_{xx}(f_2)\right)\right] = \begin{cases} \Gamma^2(f_1), & f_1=f_2 \\ 0, & \text{otherwise} \end{cases}$$

or in simpler terms

$$\text{Variance}\left(\hat{\Gamma}_{xx}(f)\right) \longrightarrow \Gamma_{xx}^2(f)$$

as $N \to \infty$

Bottom Line: The variance does not decrease as N increases and the periodogram estimator is inconsistent.