SPECTRUM ESTIMATION

Basic Concepts

Spectral Estimation Problem:

From a finite record of a stationary data sequence, estimate how the total power is distributed over frequency.

Applications:

machine monitoring — spectral analysis provides info about wear

economics, meteorology, astronomy — reveals "hidden periodicities"

speech analysis — synthesis/recognition

radar/sonar — source/target location

medicine — ECG and EEG analysis

Two Approaches

Classical, Nonparametric Approach —

signal under study is passed through a narrow bandpass filter, which is swept through the frequency band of interest, and the filter output power divided by the bandwidth is used as a spectrum estimate.

Parametric Approach —

postulate a model for the data, which provides a means of parameterizing the spectrum, and thereby reduces the spectrum estimation problem to that of parameter estimation.

Signal Assumptions

discrete-time — result of sampling in time or space

random — variation of signal cannot be exactly determined, but only specified in terms of statistical averages

complex-valued — complex-valued data arise in many applications such as complex demodulation

zero mean —

$$E[x(n)] = 0 \quad \text{for all } n$$

Energy Spectral Density of Deterministic Signals

Assume that the discrete-time signal $\ldots, x(-1), x(0), x(1), \ldots \equiv \{x(n)\} \equiv x$ has <u>finite energy</u>:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Then x possesses a discrete-time Fourier Transform (DTFT)

$$X(f) = \sum_{n=-\infty}^{\infty} x\,n\; e^{-j2\pi f n}$$

The inverse DTFT is

$$x\,n = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\, e^{j2\pi f n}\, df$$

The <u>energy spectral density</u> is

$$S_{xx}(f) = |X(f)|^2$$

Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xx}(f)\,df$$

proof:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xx}(f)\,df =$$

Interpretation:

For each $f \in [-\frac{1}{2}, \frac{1}{2}]$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, \phi_f(n)$$

where $\phi_f(n) = e^{-j2\pi fn}$

$$= \langle x, \phi_f \rangle \quad \text{"inner product"}$$

Thus

$$|X(f)| = |\langle x, \phi_f \rangle|$$

measures the "length" of the projection of x onto the span of ϕ_f. In loose terms, $|X(f)|$ shows how much (or how little) of x can be "explained" by ϕ_f, for a given value of f.

Autocorrelation Function:

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n-k)$$

Wiener - Khintchine Theorem:

$$S_{xx}(f) = \sum_{k=-\infty}^{\infty} r_{xx}(k)\, e^{-j2\pi f k}$$

Direct and Indirect Computation:

direct: $$S_{xx}(f) = \left| \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} \right|^2$$

indirect: $$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x^*(n)\, x(n+k)$$

$$S_{xx}(f) = \sum_{k=-\infty}^{\infty} r_{xx}(k)\, e^{-j2\pi f k}$$

Finite-Duration Signals

Usually, we observe only a finite duration of x. That is, we measure $x(n)$, $n = 0, 1, \ldots, N-1$.

Let $\tilde{x}(n) = \begin{cases} x(n), & n = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$

Then

$$S_{\tilde{x}\tilde{x}}(f) = |\tilde{X}(f)|^2$$

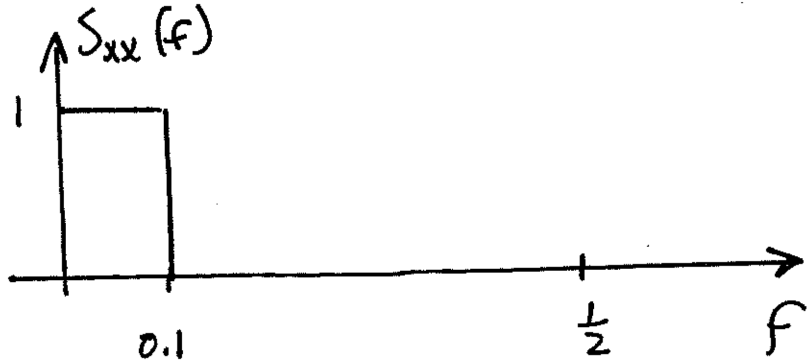
$$= |X(f) * W(f)|^2$$

Where $W(f) = \sum_{n=-\infty}^{\infty} 1_{\{0 \le n \le N-1\}} \cdot e^{-j2\pi f n}$

$$= \sum_{n=0}^{N-1} e^{-j2\pi f n}$$

Ex.

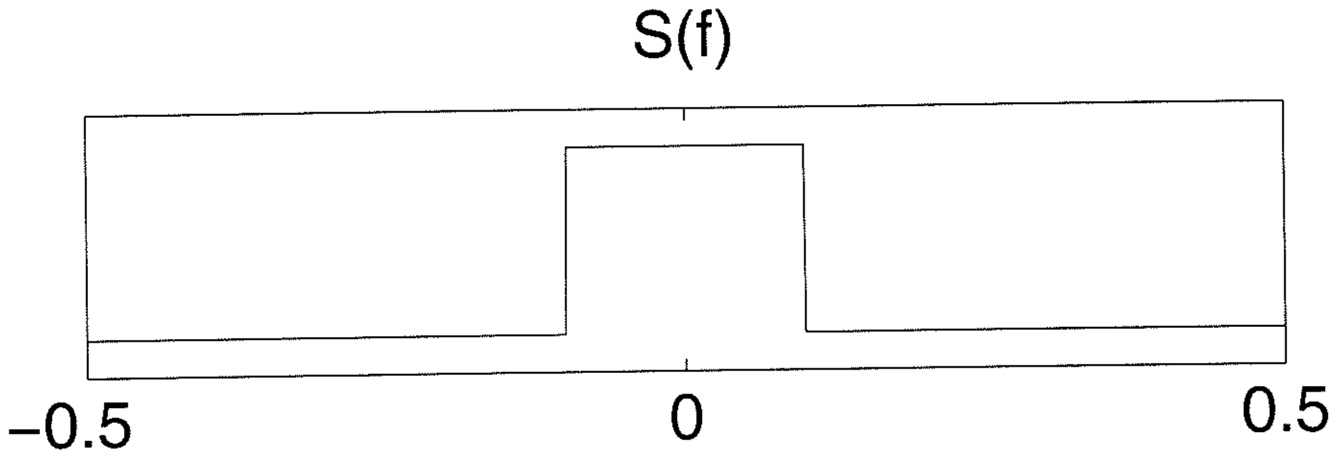
$$X(f) = \begin{cases} 1, & |f| \le 0.1 \\ 0, & \text{otherwise} \end{cases}$$

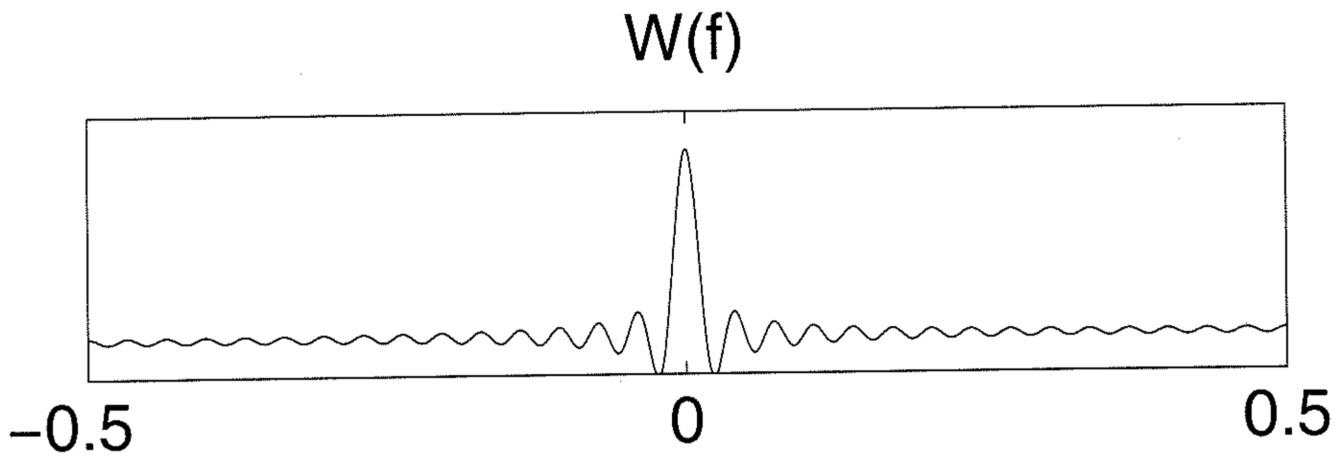


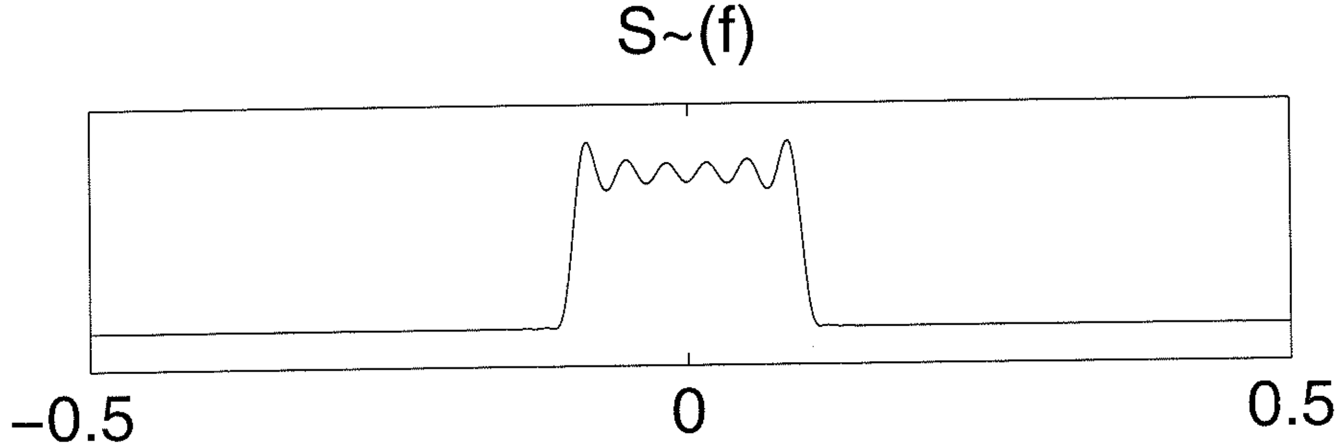
Suppose $N = 61$

$$W(f) = \sum_{n=0}^{60} e^{-j2\pi f n} = \frac{1 - e^{-j2\pi f 61}}{1 - e^{-j2\pi f}}$$

$$= \frac{e^{-j61\pi f}}{e^{-j\pi f}} \cdot \frac{\sin(61\pi f)}{\sin(\pi f)}$$


S(f)
−0.5
0
0.5


W(f)
−0.5
0
0.5


S~(f)
−0.5
0
0.5

Spectral Analysis via the FFT

$$\tilde{x}(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$S_{\tilde{x}\tilde{x}}(f) = \left| X(f) * W(f) \right|^2$$

DFT of $\tilde{x}$:

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j 2\pi \overbrace{\frac{k}{N}}^{f} \cdot n}$$

$O(N \log N)$ using FFT algorithm

Then

$$S_{\tilde{x}\tilde{x}}\left(\frac{k}{N}\right) = \left| \tilde{X}(k) \right|^2$$

$S_{\tilde{x}\tilde{x}}(f)$ sampled at $f = \frac{k}{N}$

Power Spectral Density of Random Signals

Now assume that x is a zero mean second order stationary sequence:

$$E[x(n)] = 0 \quad \text{for all } n$$

$$E[x(n)\,x^*(n-k)] \quad \text{depends only on } k, \text{ not } n$$

Covariance function:

$$r_{xx}(k) = E[x(n)\,x^*(n-k)]$$

Recall,

$$r_{xx}(k) = r_{xx}^*(-k)$$

$$r_{xx}(0) \geq |r_{xx}(k)| \quad \text{for all } k$$

Power Spectral Density:

$$\Gamma(f) = \sum_{k=-\infty}^{\infty} \gamma(k)\, e^{-j2\pi f k} \qquad (1)$$

The autocovariance can be recovered from $\Gamma(f)$ using the inverse Fourier transform:

$$\gamma(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma(f)\, e^{j2\pi f k}\, df$$

Second Definition of $\Gamma(f)$

$$\Gamma_{xx}(f) = \lim_{N\to\infty} E\left[\frac{1}{N} \left| \sum_{n=1}^{N} x(n)\, e^{-j2\pi f n} \right|^2 \right] \qquad (2)$$

This is equivalent to (1) above provided that $\gamma_{xx}(k)$ decays sufficiently fast:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} |k|\,|\gamma_{xx}(k)| = 0 \qquad (\star)$$

Equivalence of ① & ②:

$$\lim_{N\to\infty} E\left[\frac{1}{N} \left| \sum_{n=1}^{N} x(n) e^{-j2\pi f n} \right|^2 \right]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} E\left[x(n) x^*(m) \right] e^{-j2\pi f(n-m)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=-(N-1)}^{N-1} \left(N - |k| \right) \gamma_{xx}(k) e^{-j2\pi f k}$$

$$= \sum_{k=-\infty}^{\infty} \gamma_{xx}(k) e^{-j2\pi f k} + \underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{k=-(N-1)}^{N-1} |k| \gamma_{xx}(k) e^{-j2\pi f k}}_{\to 0 \text{ by } (\star)}$$

<u>Spectral Estimation Problem</u>:

From a finite-length observation $\{x(1), \ldots, x(N)\}$ of a second-order stationary random process, form an estimator $\hat{\Gamma}(f)$ of its power spectral density $\Gamma(f)$, for $f \in [-\frac{1}{2}, \frac{1}{2}]$.

The Periodogram

$$\hat{\Gamma}_p(f) = \frac{1}{N} \left| \sum_{n=1}^{N} x(n) e^{-j2\pi f n} \right|^2$$

Note similarity to

$$\Gamma(f) = \lim_{N \to \infty} E\left[\frac{1}{N} \left| \sum_{n=1}^{N} x(n) e^{-j2\pi f n} \right|^2 \right]$$

The Correlogram

Motivated by the correlation-based definition

$$\Gamma(f) = \sum_{k=-\infty}^{\infty} \gamma(k) e^{-j2\pi f k}$$

We define

$$\hat{\Gamma}_c(f) = \sum_{k=-(N-1)}^{N-1} \hat{\gamma}(k) e^{-j2\pi f k}$$

where

$$\hat{\gamma}(k) = \frac{1}{N} \sum_{n=k+1}^{N} x(n) x^*(n-k), \quad 0 \le k \le N-1$$

$$\hat{\gamma}(-k) = \gamma^*(k)$$

Claim:

$\hat{\Gamma}_c(f)$ and $\hat{\Gamma}_p(f)$ are equivalent.

(prove in homework)

Performance of the Periodogram

In short, the performance of the periodogram (and correlogram) is poor. There are two reasons for this:

1) estimation errors in $\hat{\gamma}(k)$ are on the order of $\frac{1}{\sqrt{N}}$ for large N. Because $\hat{\Gamma}_p(f)$ is the sum of $2N-1$ such estimates, there is no guarantee that the total errow will die out as N increases.

2) If $\gamma(k)$ decays slowly, then the periodogram estimates will be biased.

Even though the periodogram possesses these undesirable properties, a careful understanding of its performance will help us design improved nonparametric spectral estimators.

MSE, Bias and Variance

Let u be a quantity to be estimated and let $\hat{u}$ be an estimator (random variable).

The mean squared error (MSE) is:

$$
\begin{aligned}
\text{MSE} &\equiv E\left[|\hat{u} - u|^2\right] \\
&= E\left[|\hat{u} - E[\hat{u}] + E[\hat{u}] - u|^2\right] \\
&= \underbrace{E\left[|\hat{u} - E[\hat{u}]|^2\right]}_{var(\hat{u})} + \underbrace{|E[\hat{u}] - u|^2}_{|bias(\hat{u})|^2} \\
&\quad + 2\,\text{Re}\left(E\left[\hat{u} - E[\hat{u}]\right] \cdot \left(E[\hat{u}] - u\right)\right)
\end{aligned}
$$

By separately considering the bias and variance components of the MSE, we gain insight into the source of error and in ways to reduce the error.

Next, we will analyze the bias and variance of the periodogram estimator $\hat{\Gamma}_p(f)$. This analysis will point us towards improved spectral estimators.