Signal Analysis & Processing

The first step in many scientific and engineering problems is often signal analysis.

Given measurements or observations of some physical process, we ask the simple question

"What have we got here?"

Some have called this step "data exploration". We search for structure and features in the data to help us understand the phenomenon at hand and to guide subsequent processing actions.

Data Exploration:

In our search for structure in the data, it is often helpful to <u>transform</u> the data into an alternate representation domain in which features of interest are more easily detectable.

<u>Ex.</u> Suppose we are interested in periodic behavior in a time-sequence of measurements.

Fourier Analysis

Fourier analysis is one of the major achievements in physics and mathematics. It is central to signal theory and processing for several reasons.

1. Sinusoidal Basis Functions

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{-j 2\pi f_k \cdot t}$$

Fourier Series

Applications

- analysis of physical waves (acoustics, vibrations, geophysics, optics)
- analysis of periodic processes (economics, biology, astronomy)

2. Fourier Transform and Linear Filtering

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}$$

Fourier Transform

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi f t}\, df$$

Convolution in time $\equiv$ multiplication in frequency

3. Fourier Analysis of Stationary Gaussian Processes

Suppose $X(t)$ is a stationary, zero-mean, Gaussian random process. Then $X(t)$ can be represented as a white noise process passed through a linear, time-invariant filter.

$$\underbrace{w(t)}_{\text{white noise}} \longrightarrow \underbrace{\boxed{H}}_{\text{LTI filter}} \longrightarrow X(t)$$

Moreover, a Gaussian process is completely described by its mean and covariance functions.

4. Pragmatic Issues

- The important features of Fourier analysis have led to the development of a large number of algorithms and implementations for frequency analysis.

- Most notably, perhaps, is the Fast Fourier Transform algorithm, which enables the calculation of the Fourier series of an N-point sequence in just $O(N \log N)$ operations (compared to $O(N^2)$ in direct calculation).

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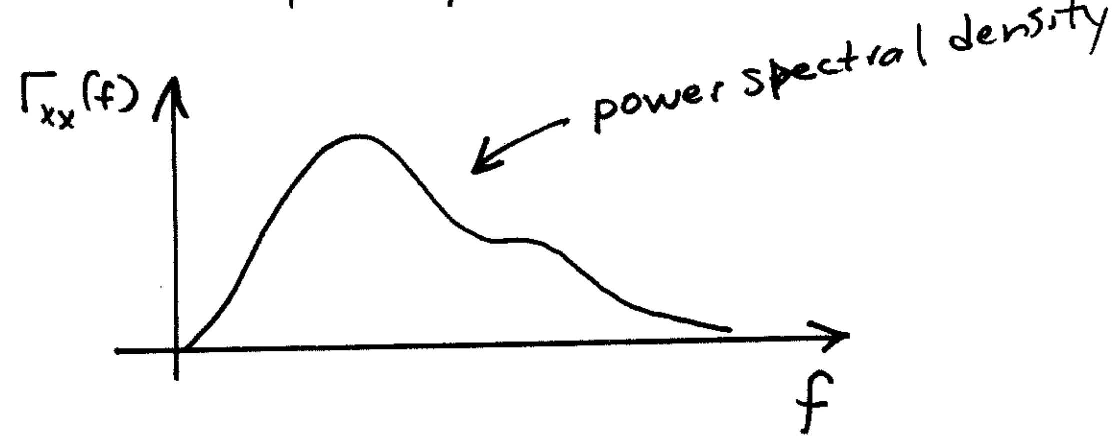
Spectrum Estimation

The great importance of Fourier analysis leads one to the study of spectrum estimation.

Roughly speaking, the problem is simply to determine the amount of energy in a signal at each frequency



The power spectral density quantifies the distribution of signal energies across frequency.

Many of the phenomena that we are interested in studying are random in nature. For example, in radar and sonar subtle variations in the transmission medium are very unpredictable and are well modeled as random processes.

Due to random fluctuations, we will adopt a statistical approach to spectrum estimation. The basic problem is to determine the power spectral de ity of a signal given measurements

$$x(n), \quad n = 0, \ldots, N-1$$

Limitations of Fourier Analysis

- The inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

reveals that any value $x(t)$ of the signal at <u>one</u> time instant can be regarded as an infinite superposition of complex exponentials, or <u>everlasting</u> and completely nonlocal waves.

- Even though this mathematical representation can aid us in the discovery of signal structure in certain cases (e.g., periodic behavior), it can also distort the physical reality.

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Transient Signals

Suppose that the signal $x(t)$ is exactly zero outside a certain time interval (e.g., by switching a machine on and off).

Although this signal can still be studied by Fourier techniques, the frequency domain representation has a very artificial behavior. The time signal's zero values are achieved by an infinite superposition of virtual waves that interfere in such a way that they cancel each other out.

We can list some disadvantages of Fourier analysis.

1. Many signals, especially those which are transient in nature, are not well represented in terms of everlasting sinusoidal waves.

<u>Ex.</u> Images contain many edges that cannot be efficiently represented with sinusoids.

2. Often, it is the non-stationary or transient components of a signal that carry the <u>important</u> information.

<u>Ex.</u> Imagine that you are on a beach watching the waves roll in to shore. Your peaceful state is broken as a porpoise leaps from the water!

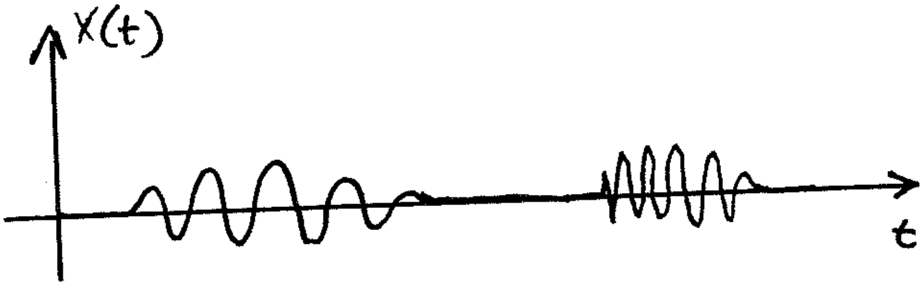
3. As we have seen, Stationary Gaussian processes are intimately linked with Fourier analysis. However, many signals we are interested in (e.g., speech, images) are not well modeled as stationary <u>and</u> Gaussian.

In many cases, the key information and structure in a signal is not conveyed by the <u>global</u>, <u>nonlocal</u> frequency content, but rather by the local variations in frequency over time (or space in the case of images). This suggest a study of

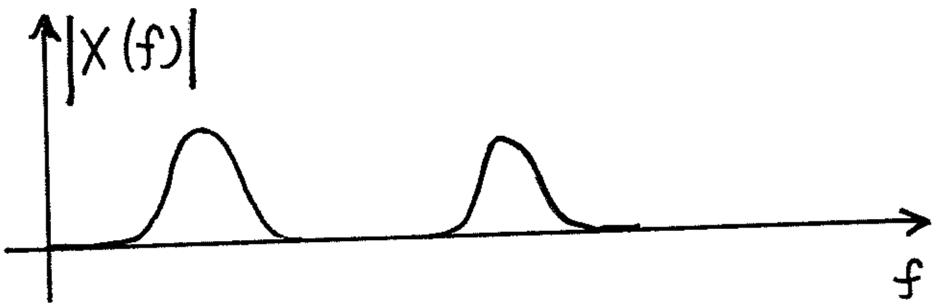
joint <u>time-frequency analysis</u>.

Time - Frequency Wedding

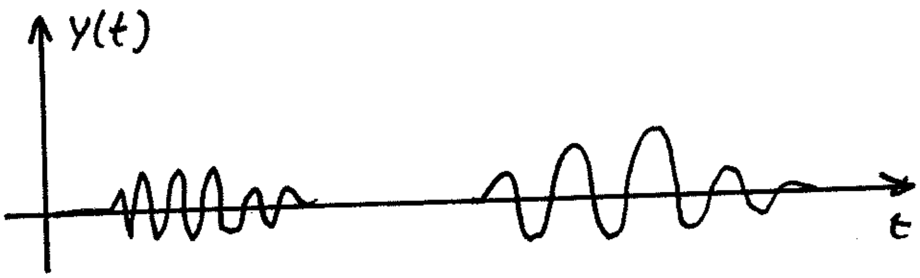
- Consider the following time signal



and the magnitude of its fourier transform



- From the magnitude alone, we are unable to distinguish $x(t)$ from $y(t)$



That is, $|X(f)| = |Y(f)|$

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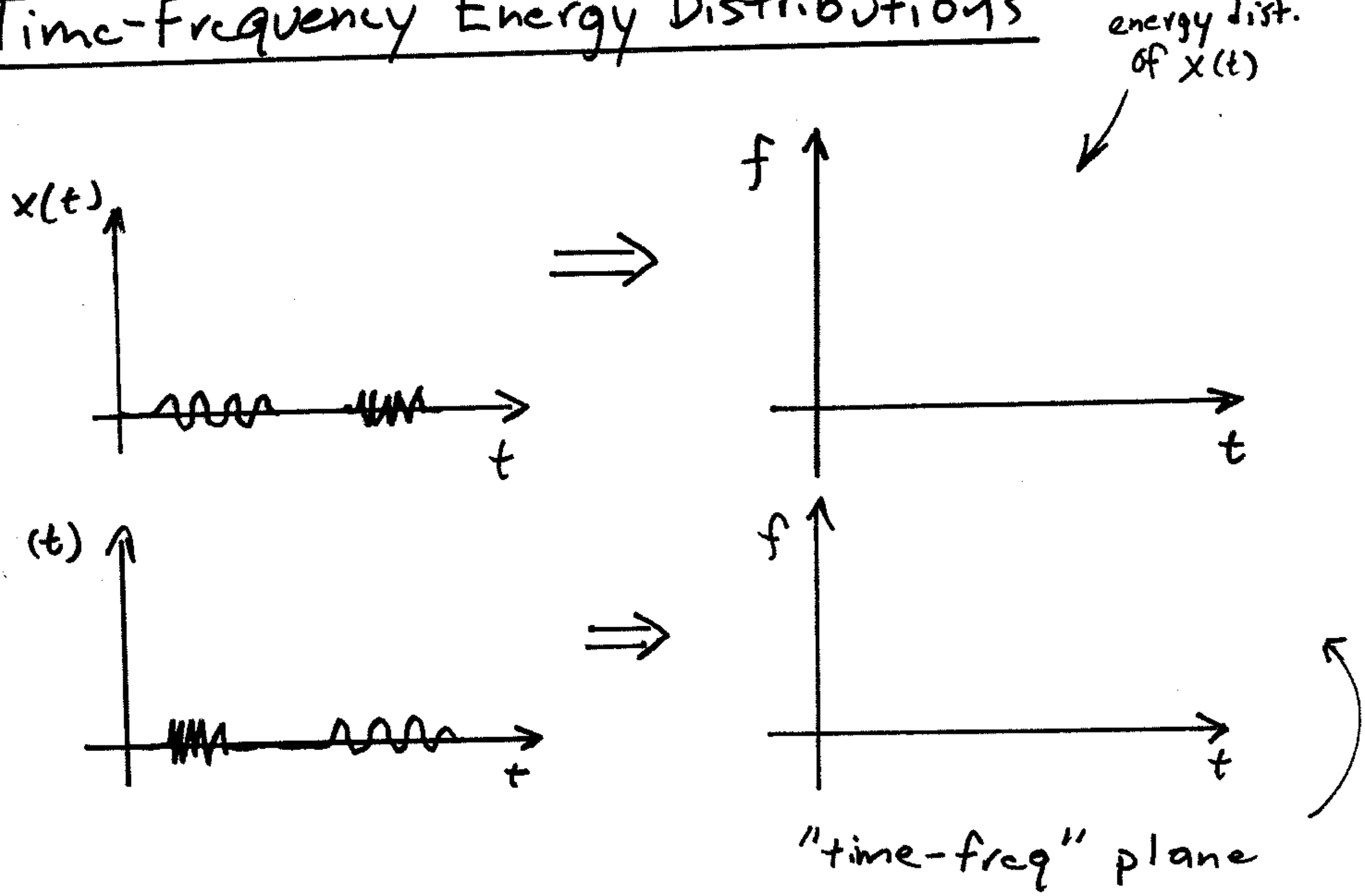
The magnitude (or energy) of the Fourier transform does not tell us <u>when</u> the high frequency and low frequency components occur.

Time information is "lost" in the phase.

Therefore, the Fourier transform is not effective as a method for studying the <u>time-frequency energy distribution</u>.

In many situations, music for example, it is crucial to know <u>when</u> different frequency components occur.

Time-Frequency Energy Distributions



What sort of analysis will provide us with a time-frequency energy distribution?

What kind of basis functions (not sinusoids!) will enable such an analysis?

Time-Frequency Analysis

Key Idea: (1946)

A phycisist, Gabor, defined elementary time-frequency "atoms" as waveforms that have minimal spread in the time-frequency plane.

Define

$$g_{t_0,\omega_0}(t) \equiv \underbrace{w(t-t_0)}_{\substack{\text{window function} \\ \text{(time-localized)}}} e^{i\omega_0 t}$$

complex sinusoid @ radian frequency $\omega_0 = 2\pi f_0$

This is simply a windowed sinusoid.

The Fourier transform of g_{t_0,ω_0} is

$$G_{t_0,\omega_0}(\omega) = \int_{-\infty}^{\infty} g_{t_0,\omega_0}(t)\, e^{-j\omega t}\, dt$$

$$=$$

The energy of g_{t_0,ω_0} is concentrated

<u>in time</u> on the interval of width σ_t about t_0

$\sigma_t \equiv$ standard deviation of $|W(t)|$

<u>in frequency</u> on the frequency band of width σ_ω centered at ω_0

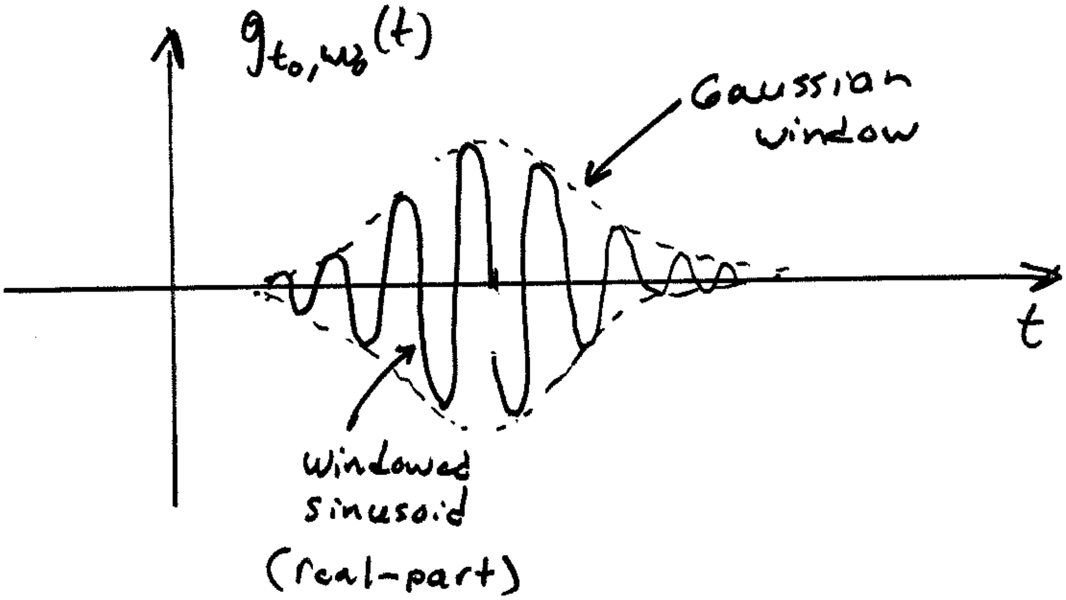
$\sigma_\omega \equiv$ standard deviation of $|W(\omega)|$

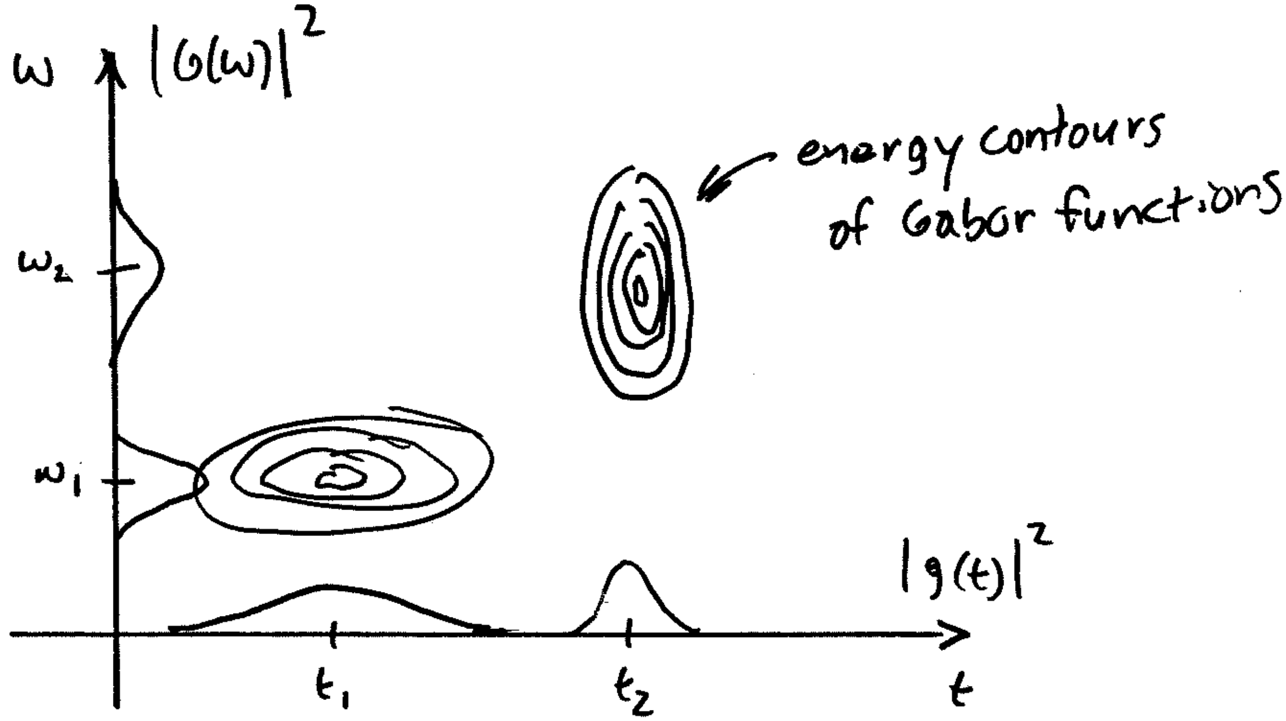
The product $\sigma_t \cdot \sigma_\omega$ measures the area in the time-frequency plane in which most of the energy

$$\int g_{t_0,\omega_0}^2(t)\, dt$$

is concentrated.

$\sigma_t \cdot \sigma_w$ is minimized when $w(t)$ is a Gaussian function, in which case $g_{t_0,\omega_0}(t)$ are called <u>Gabor functions</u>.





Time-Frequency Energy Distributions

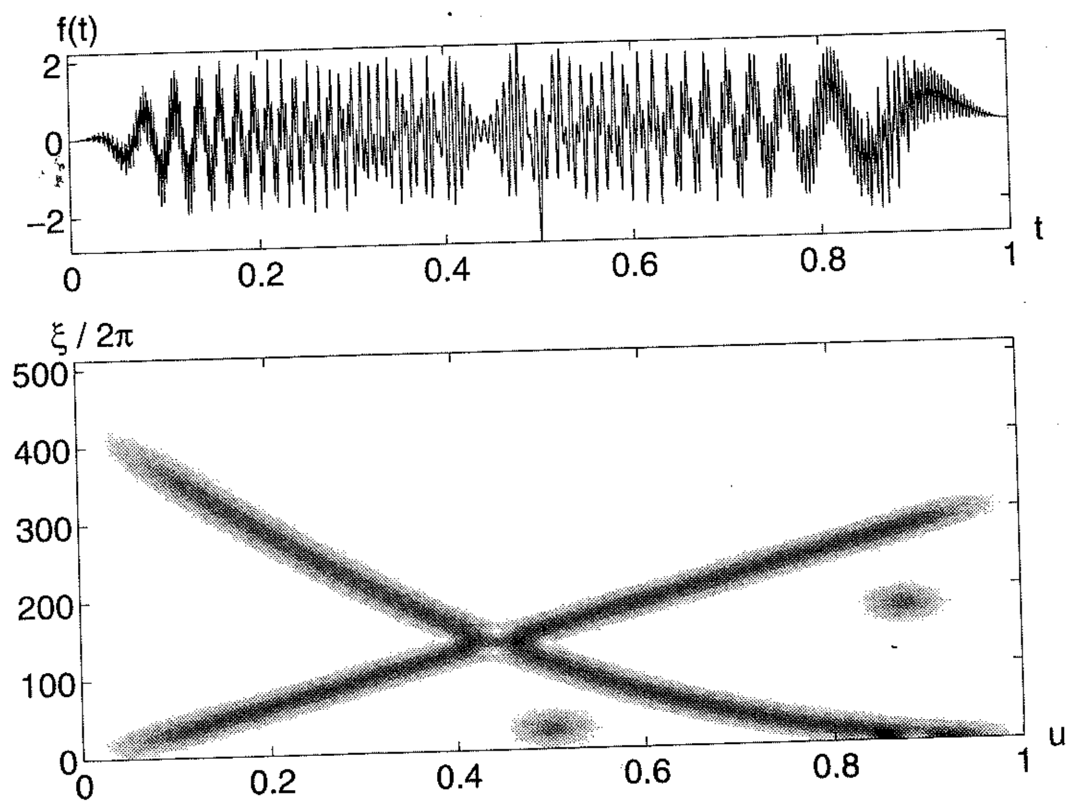
Let $x(t)$ be a finite-energy time signal. Then

$$|S_x(t_0,\omega_0)|^2 = \left| \int_{-\infty}^{\infty} x(t)\, w(t-t_0)\, e^{-j\omega_0 t}\, dt \right|^2$$

measures the energy in the signal at time t_0 and frequency ω_0.

$|S_x(t_0,\omega_0)|^2$ as a function of (t_0,ω_0) is called the <u>spectrogram</u> of $x(t)$.

Ex. Spectrogram



signal is composed of

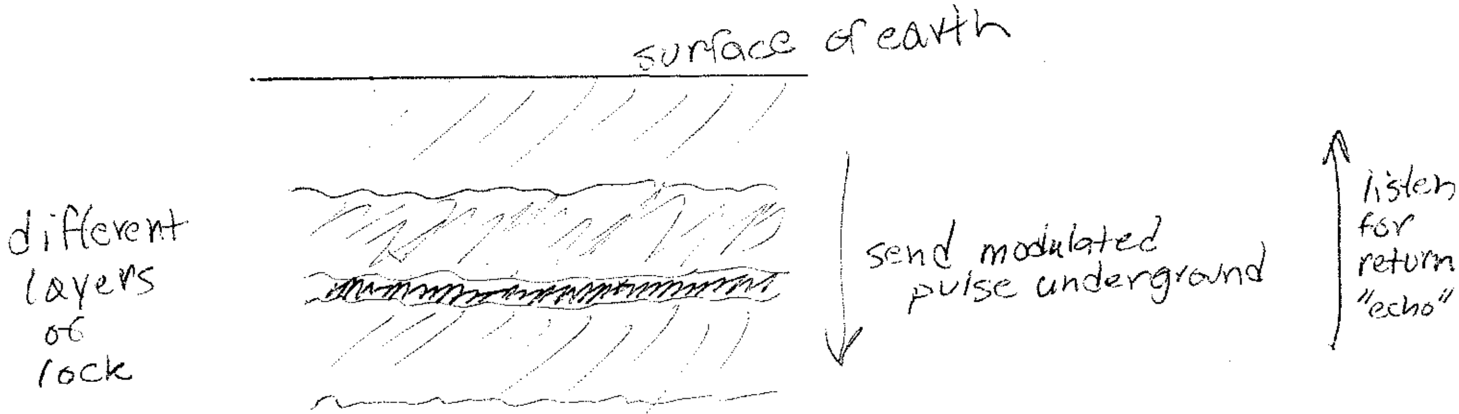
a linear "chirp"

a quadratic chirp

two modulated Gaussians

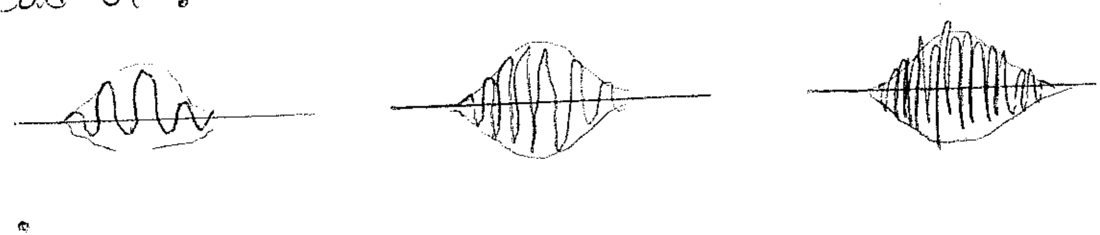
Key Idea:

(1975) Morlet, a seismologist in France, knew that modulated pulses (e.g., Gabor fncs) had too long of a time duration at high frequencies to separate closely space layers of rock.

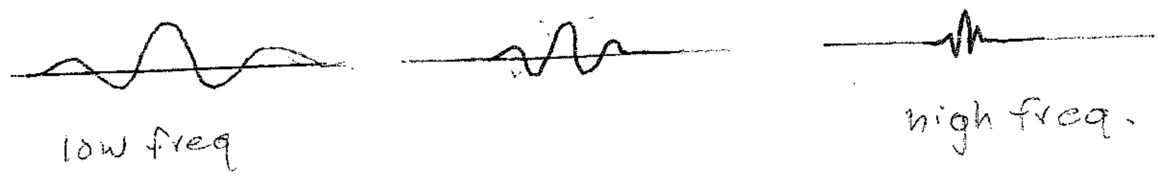


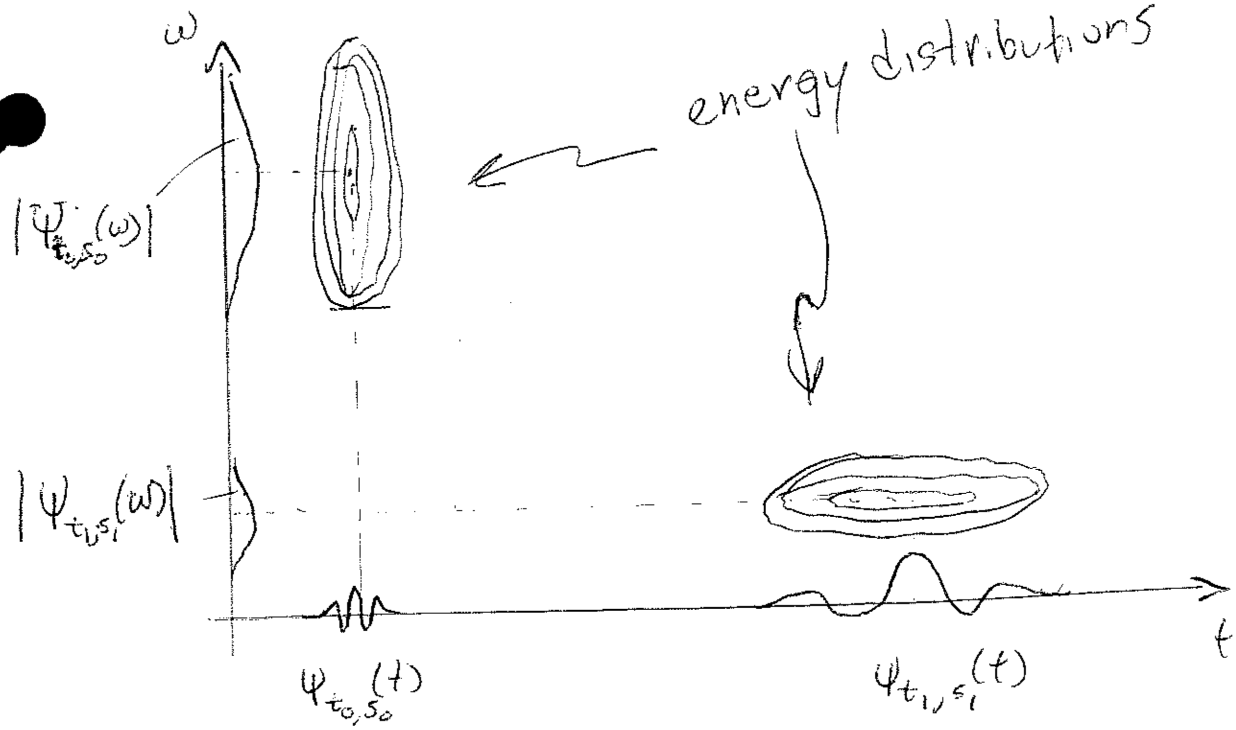
He thought of sending shorter waveforms at higher freqs to avoid this problem

Instead of:



Use:





$$\Psi_{s_0,t_0}(t) = \frac{1}{\sqrt{s_0}}\,\Psi\left(\frac{t-t_0}{s_0}\right)$$

where $\int_{-\infty}^{\infty} \Psi(t)\,dt = 0$ (must oscillate)

t_0 is time "center"

s_0 is <u>scale</u> (proportional to freq)

Ψ is called a <u>wavelet</u>.

Wavelets

A wavelet, as the name suggests, is a small (localized) wave function.

sinusoid

wavelet

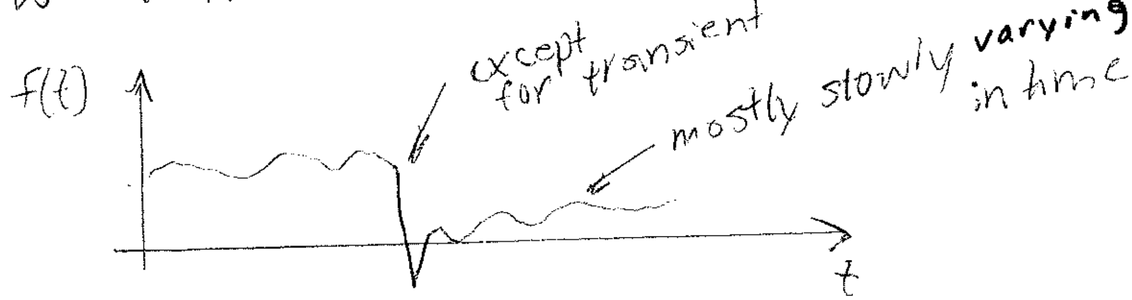
Continuous Wavelet Transform

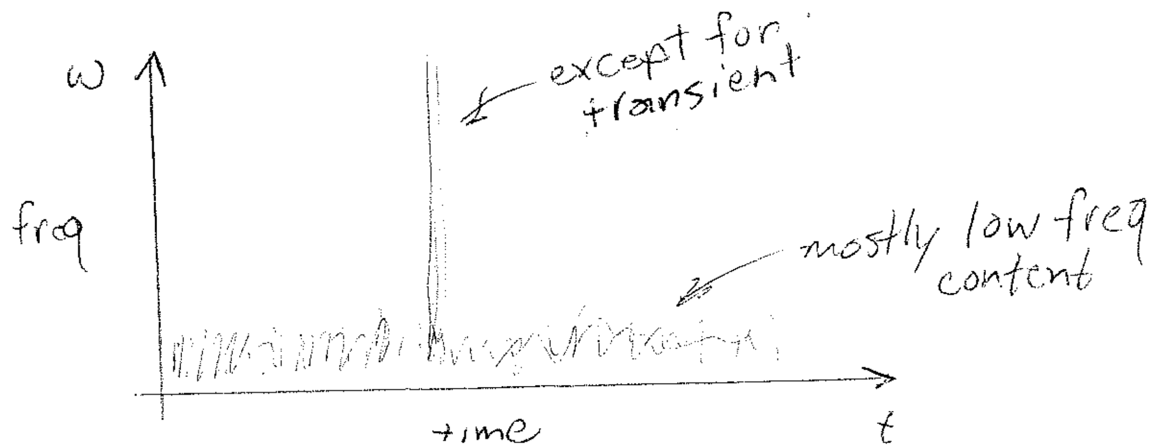
$$W_x(t_0, s_0) = \int_{-\infty}^{\infty} x(t)\, \psi_{s_0,t_0}(t)\, dt$$

$|W_x(t_0, s_0)|^2$ provides another means of studying the time-frequency energy density of $x(t)$.

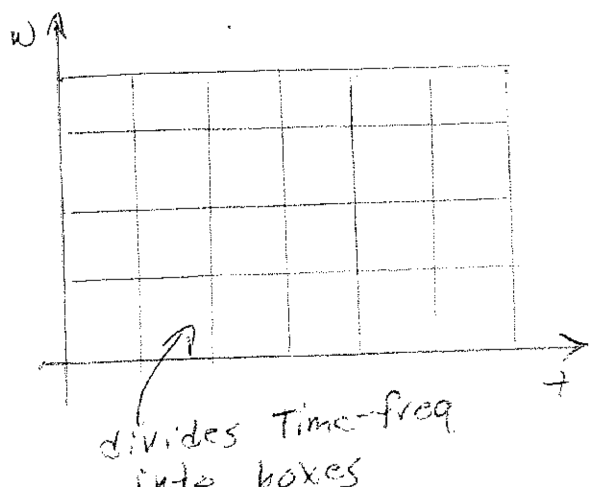
Multiscale Analysis

The <u>multiscale</u> (as opposed to multifreq) analysis provided by wavelets is especially well suited to signals with transients or

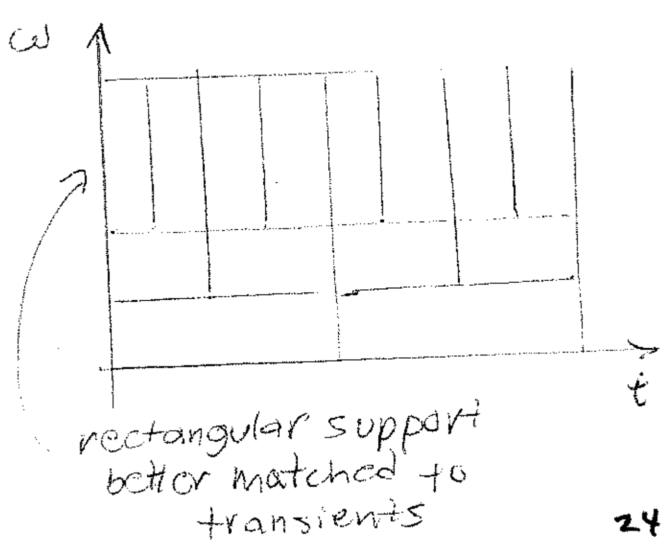




Gabor Analysis (idealized)



"Wavelet Analysis" (idealized)

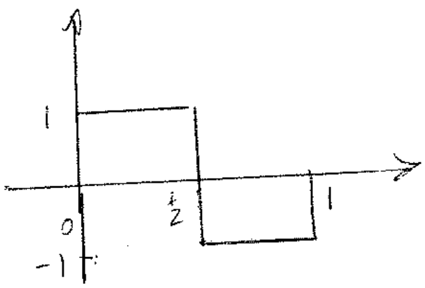


Signal Representation with Wavelets

Key Idea: Haar (1908)

One can construct a simple piecewise function

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



"Haar" wavelet

whose dilations and translations generate an orthonormal basis of $L^2(\mathbb{R})$:

$$\left\{ \psi_{j,k}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j k}{2^j}\right) \right\}_{(j,k) \in \mathbb{Z}^2}$$

Any finite energy signal f can be decomposed over this (orthogonal) wavelet basis

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k} \quad , \quad \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)d$$

(note similarity to Fourier series)

Since $\psi(t)$ has a zero average, each partial sum

$$d_j(t) = \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}$$

can be interpreted as "detail" variations in f at scale 2^j.

If f has smooth variations, then the approximation at scale 2^J:

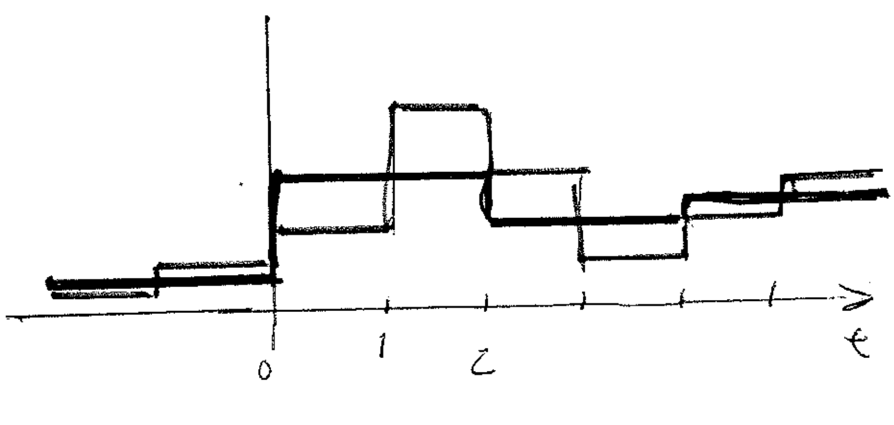
$$f_J(t) = \sum_{j=J}^{\infty} d_j(t) \approx f(t)$$

Connection to DSP

Suppose we have an f and a (Haar) approximation f_J

f_{J+1} is a even coarser approximation

f_{J+1} can be easily expressed in terms of f_J



ex. $f_{J+1}(1) = \left(f_J\left(\frac{1}{2}\right) + f_J\left(\frac{3}{2}\right)\right)/2$

$\Rightarrow$ values of f_{J+1} approximation are just digital filtered $\left(\frac{1}{2}, \frac{1}{2}\right)$ versions of f_J values (wavelets = filter banks)

Limitation of Haar Wavelets

Piecewise constant
$\Rightarrow$ irregular, "blocky"
approximations

Are there generalizations to Haar analysis that circumvent this problem?

Yes

Key Idea: Strömberg (1980)
Piecewise linear ψ
provides better approximations for many classes of signals.

Key Idea: Meyer (1986), Mallat (1989)

Multiresolution / multiscale analysis formally derived in a more mathematical setting. Multiresolution analysis viewed as an expansion in terms of orthogonal basis functions (wavelet basis)

Key Idea: Daubechies (1988)

First highly practical wavelet bases are introduced. Fast implementations using digital filter banks.

Key Idea: Meyer (1986), Frazier & Jawerth (1985)

Wavelet bases are very efficient representation systems for wide classes of signals. Wavelets are unconditional bases for many function spaces.

Key Idea: Donoho (1993)

Wavelets are (in a certain sense) optimal bases for signal compression, estimation and recovery

Key Idea: Filter Banks (DSP connection)

Croisier, Esteban, Galand (1976)
Smith and Barnwell, Vaidyanathan
and Vetterli (1980's)

* Complete theory for decomposing a signal in subsampled components with a filtering scheme and perfectly reconstructing original signal frome these components.

* Based on quadrature mirror filters. (QMFs)

* Multiresolution theory of wavelets (Mallat) proves that any QMF Characterizes a wavelet that generates an orthonormal basis of $L^2(\mathbb{R})$.

Multiscale Analysis using Discrete Wavelet Transform

high resolution image

$\Longrightarrow$

mid resolution + prediction errors

$\Longrightarrow$

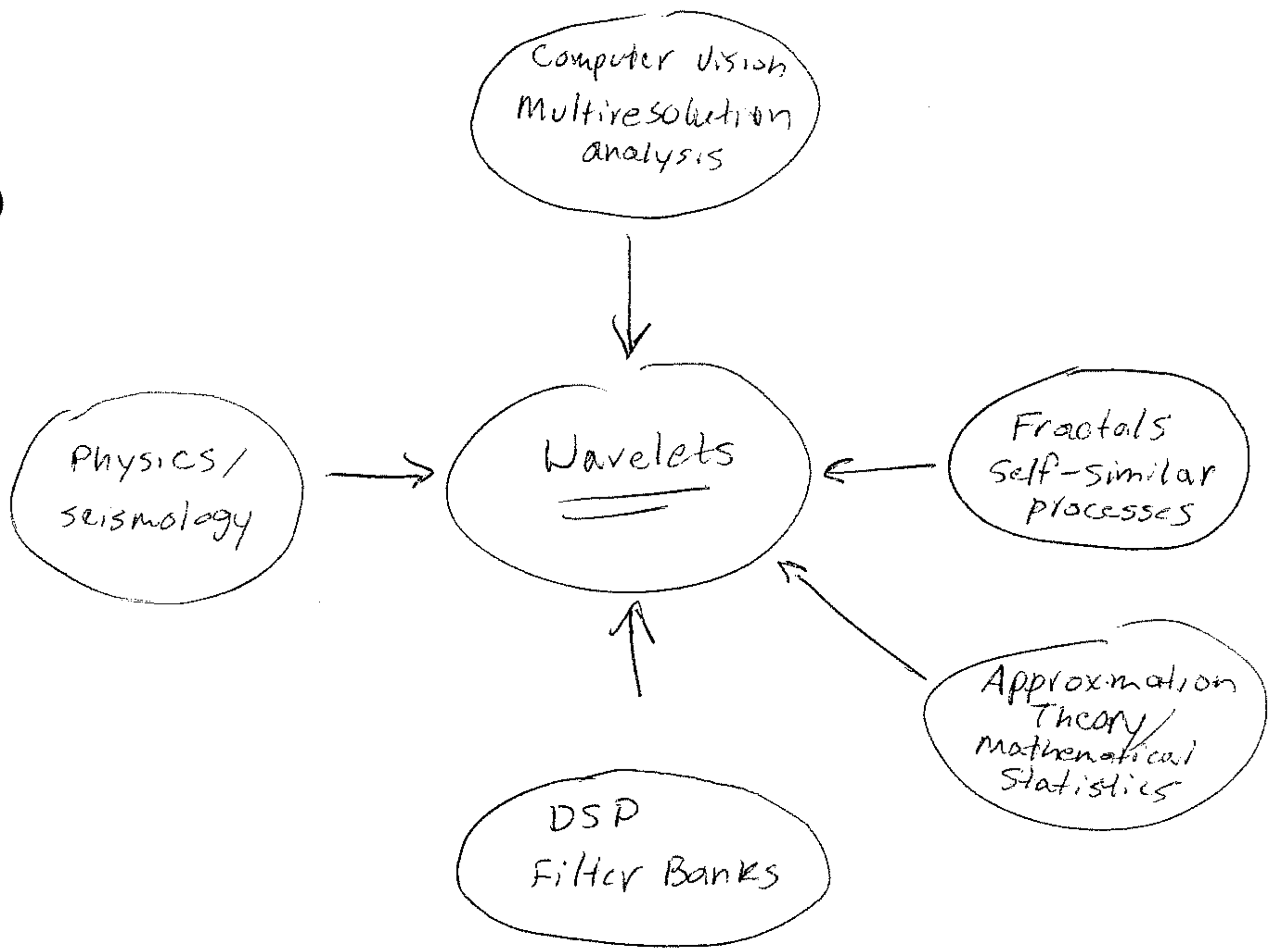
low resolution + prediction errors

prediction errors $\equiv$ wavelet coefficients

prior information $\Rightarrow$ most coefficients are zero

Key Idea Mandelbrot (1982)

Multiscale zooming capability of the wavelet transform not only helps to locate isolated singularities, but can also characterize more complex fractal signals.



To Come

Spectral Estimation (~ 2 weeks)

Frequency energy densities

Time-frequency Analysis (~ 2 weeks)

Time-frequency energy densities

Introduction to Wavelets (~ 1 week)

Basic multiscale analysis

The DWT (~ 1 week)

discrete wavelet transform
theory and methods

Signal Representations (~ 1 week)

Bases, Frames, expansions

Wavelet Bases (~ 2 weeks)

Construction and Design

Approximations with Wavelets (~2 weeks)

Basic approximation theory

Signal Processing with Wavelets (~ 2 weeks)

compression, denoising,
statistical modeling and analysis
applications in image processing

Image Analysis - Beyond Wavelets
(~1 week)