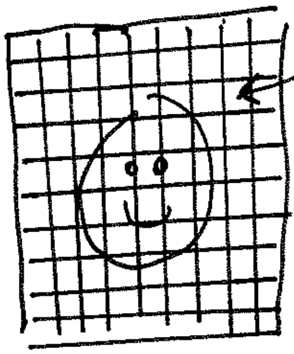
Application to Image Restoration



row index

column index

$x[n_1, n_2]$

(n_1, n_2) pixel of image

image samples $\equiv$ pixels

2-d Convolution

$$x[n_1,n_2] * g[n_1,n_2]$$

$$= \sum_{k_1} \sum_{k_2} g[k_1,k_2]\, x[n_1-k_1, n_2-k_2]$$

Ex.

$$g[n_1,n_2] = \begin{cases} \frac{1}{4}, & n_1, n_2 = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

$\Longrightarrow$ $x[n_1,n_2] * g[n_1,n_2]$ results in 4-pixel averaging ("smoothing") of original image $x[n_1,n_2]$.

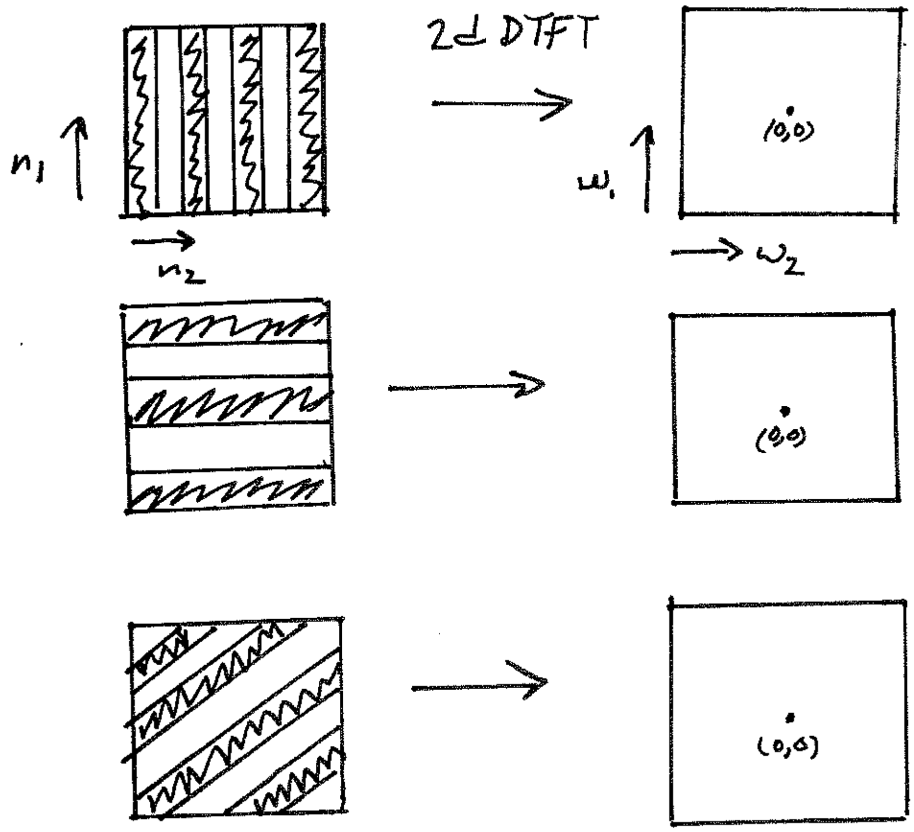
2-d DTFT

$$X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x[n_1, n_2]\, e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

freq in row direction (ω_1)

freq in column direction (ω_2)

Ex.



$$g[n_1, n_2] * x[n_1, n_2] \overset{\text{DTFT}}{\longleftrightarrow} G(\omega_1, \omega_2)\, X(\omega_1, \omega_2)$$

2-d DFT

$x[n_1,n_2]$ $N \times N$ image

$$X[k_1,k_2] = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} x[n_1,n_2]\, e^{-j\left(\frac{2\pi k_1 n_1}{N} + \frac{2\pi k_2 n_2}{N}\right)}$$

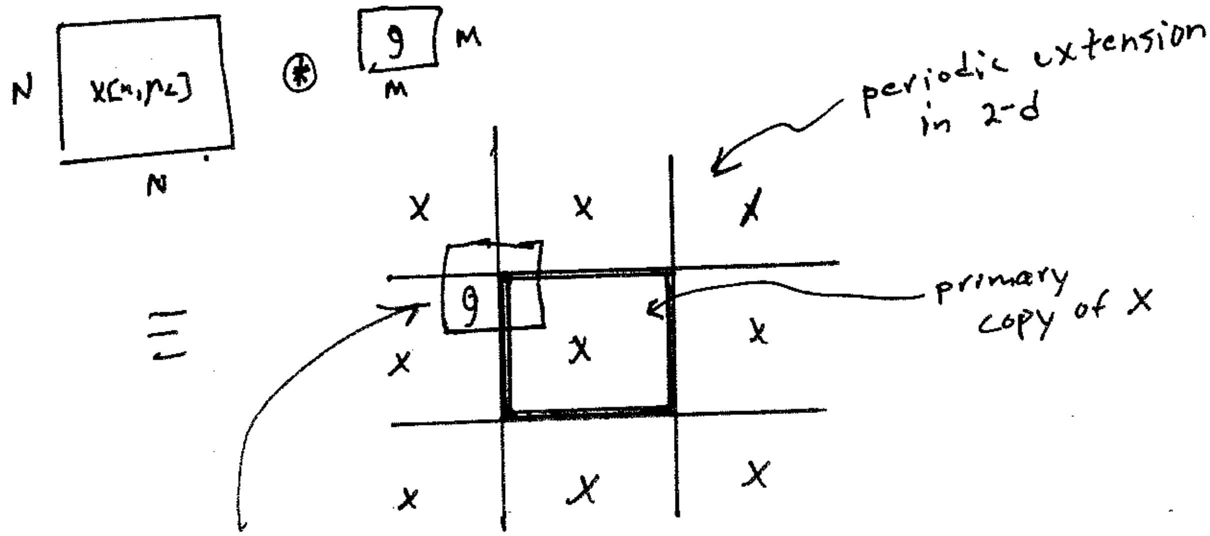
$$k_1, k_2 = 0, \ldots, N-1$$

$$x[n_1,n_2] = \frac{1}{N^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} X[k_1,k_2]\, e^{j\frac{2\pi k_1 n_1}{N}}\, e^{j\frac{2\pi k_2 n_2}{N}}$$

$$G[k_1,k_2] \cdot X[k_1,k_2] \xleftrightarrow{\text{DFT}} g[n_1,n_2] \circledast x[n_1,n_2]$$

($\circledast$: 2-d circular convolution)

2-d Circular Conv:



reg. convolve g with periodically extended version of x

→ distortion at boundary of image.

2-d FFT:

$x[n_1, n_2]$ is $N \times N$ image

Note: DFT is separable in rows and columns

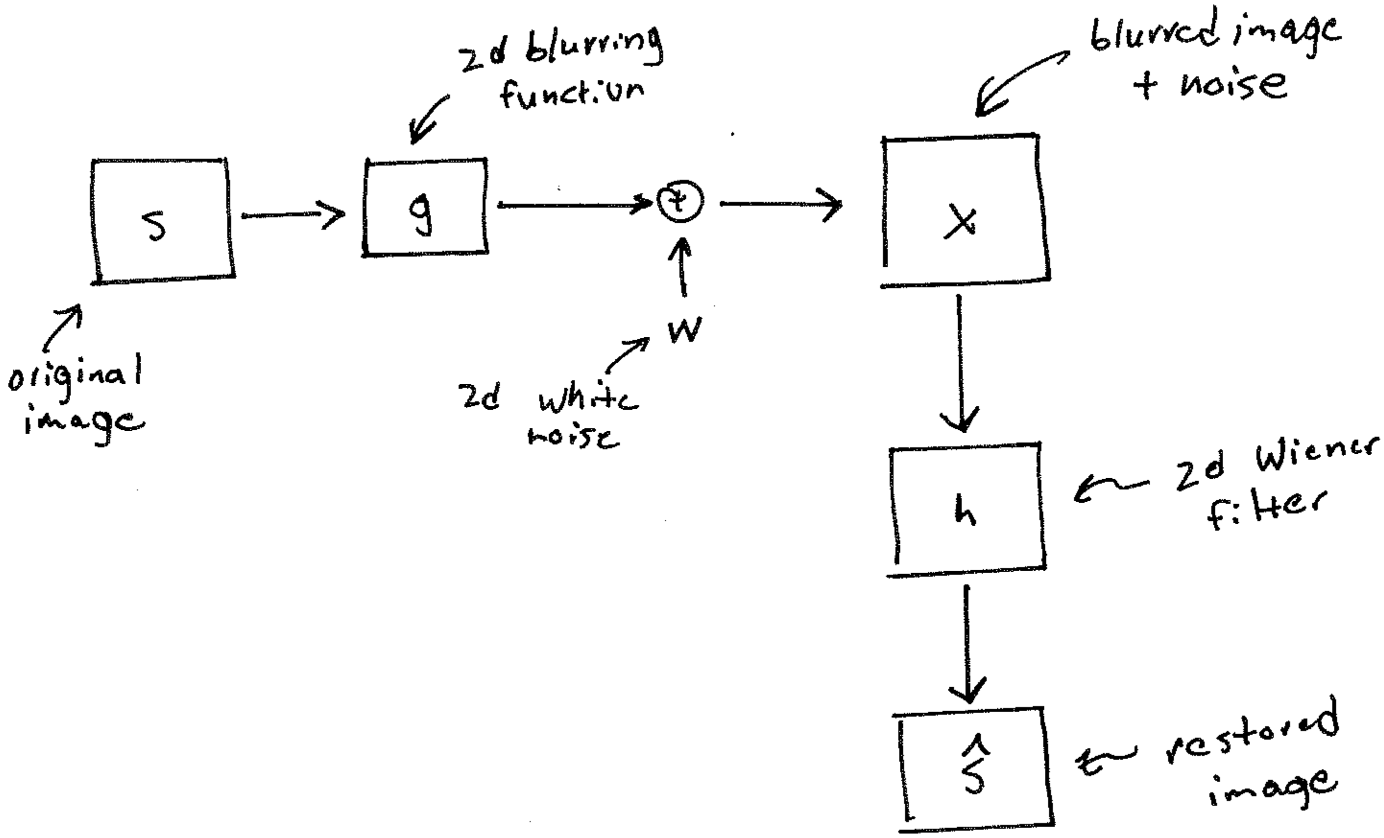
$$X[k_1, k_2] = \sum_{k_1=0}^{N-1} \sum_{n_2=0}^{N-1} x[n_1, n_2] \, e^{-j\frac{2\pi k_1 n_1}{N}} \, e^{-j\frac{2\pi k_2 n_2}{N}}$$

$$= \underbrace{\sum_{n_1=0}^{N-1} e^{-j\frac{2\pi n_1 k_1}{N}} \underbrace{\sum_{n_2=0}^{N-1} x[n_1, n_2] \, e^{-j\frac{2\pi k_2 n_2}{N}}}_{\text{1-d FFTs of each column}}}_{\text{then 1-d FFT along each row}}$$

Thus, using 1-d FFTs, the 2-d DFT can be computed in

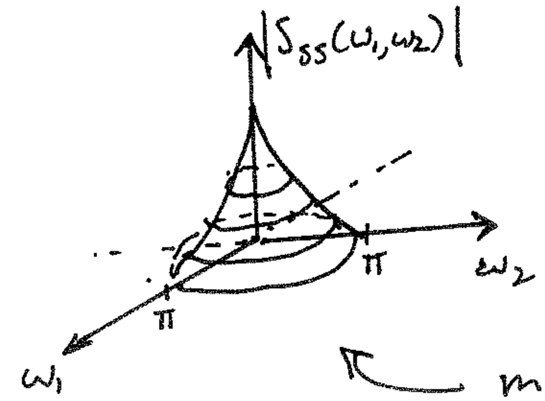
$O(N^2 \log N^2)$ operations

Image Deblurring:



Model image as 2-d random process:

$$S_{ss}(\omega_1, \omega_2) = \text{radially symmetric 2-d function}$$



mostly low freq energy (images mostly smooth) but some high freqs, too (edges)

2-d Wiener Filter

$$H(\omega_1,\omega_2) = \frac{G^*(\omega_1,\omega_2)\, S_{ss}(\omega_1,\omega_2)}{|G(\omega_1,\omega_2)|^2 S_{ss}(\omega_1,\omega_2) + S_{ww}(\omega_1,\omega_2)}$$

Sample

$$\Rightarrow \quad H[k_1,k_2] = H\left(\frac{2\pi k_1}{N}, \frac{2\pi k_2}{N}\right)$$

$$k_1, k_2 = 0, \ldots, N-1$$

DFT Implementation:

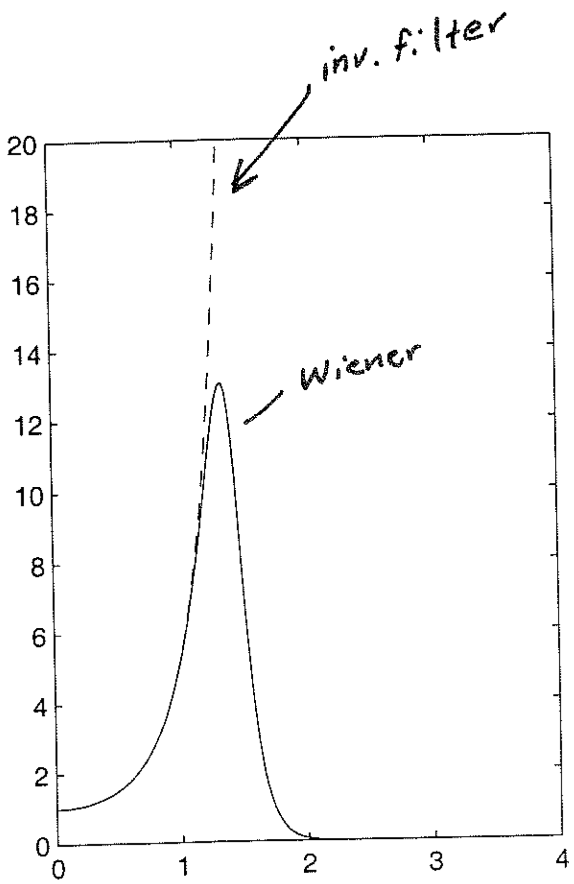
$$\tilde{S}[k_1,k_2] = H[k_1,k_2] \cdot X[k_1,k_2]$$

fast!! $O(N^2 \log N^2)$

equivalent to 2-d circular conv. in space

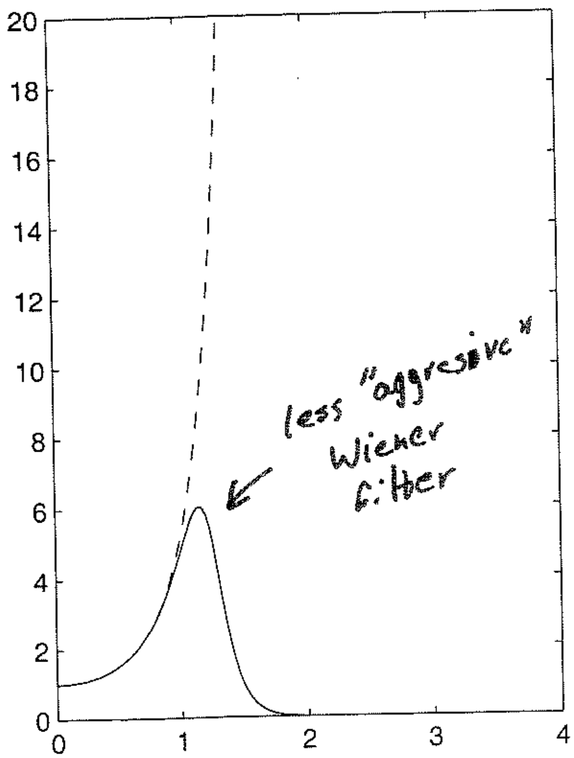
some possible distortion esp. near boundary of image

Ex. Blurring with little noise



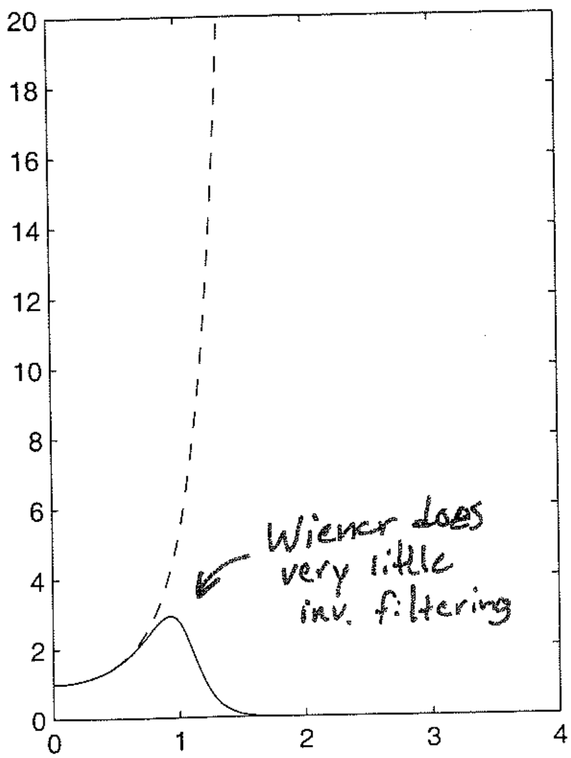
Blurring with moderate noise

noise "artifact"

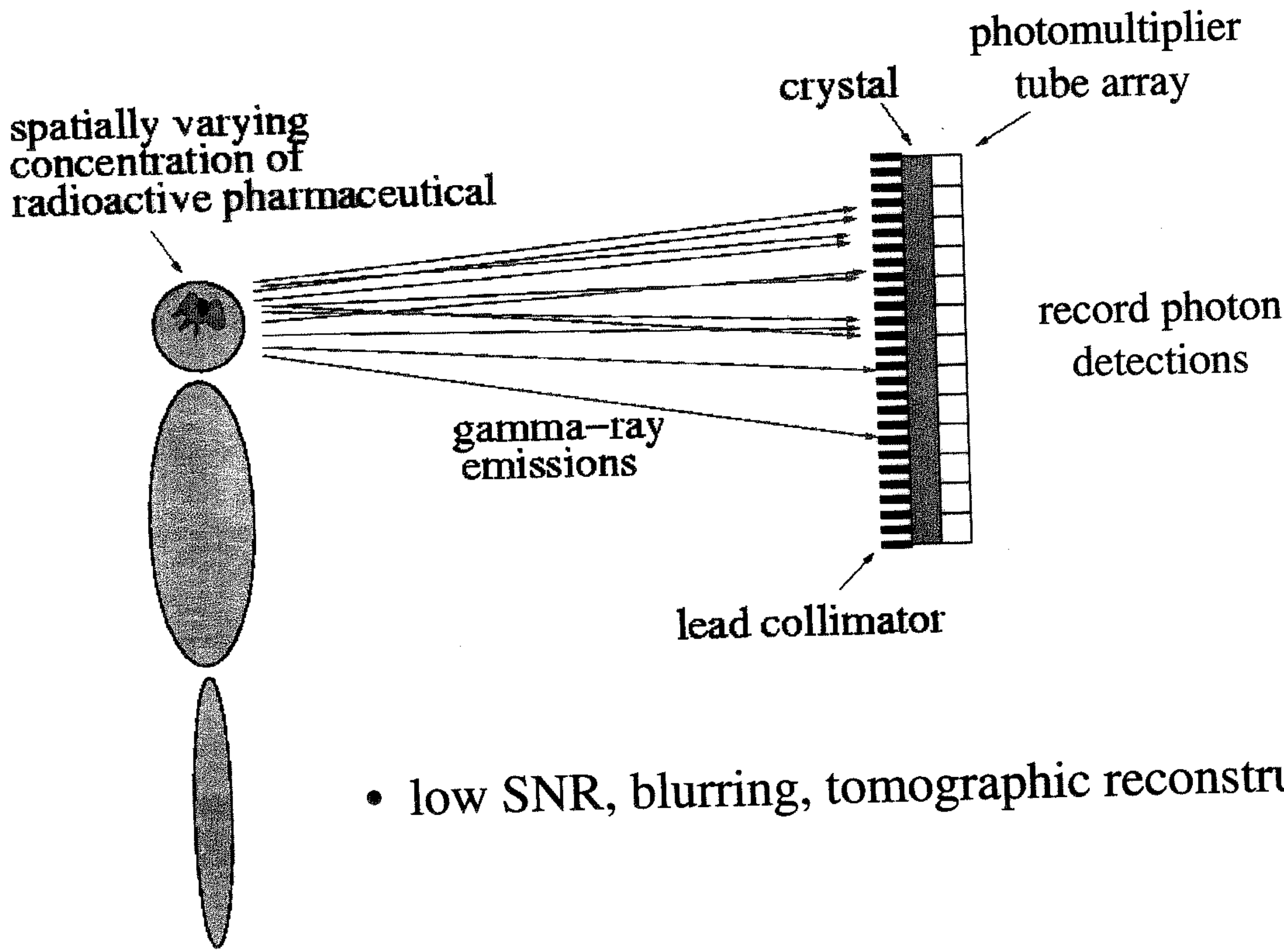


Blur with heavy noise

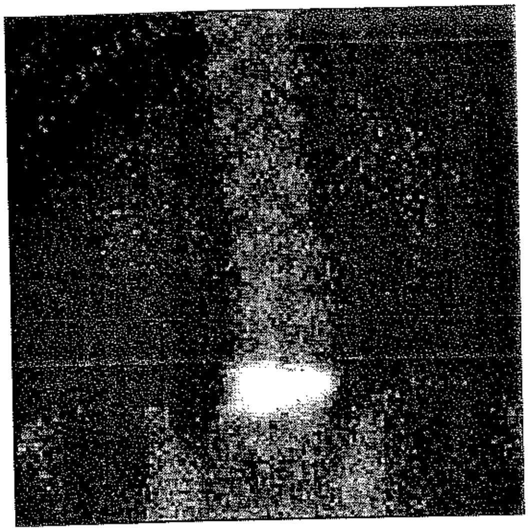
severe noise artifacts



Nuclear Medicine Imaging

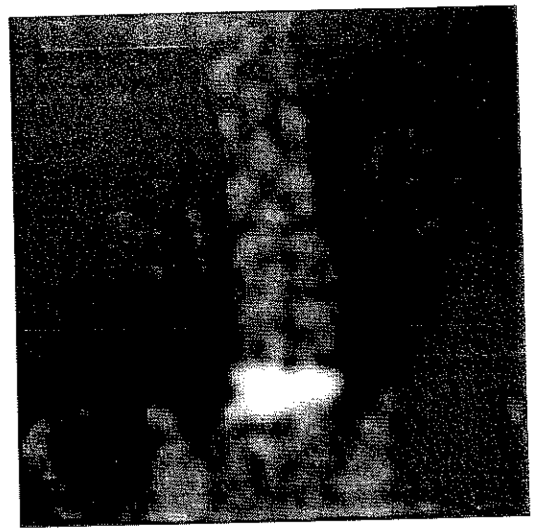


- low SNR, blurring, tomographic reconstruction raw image



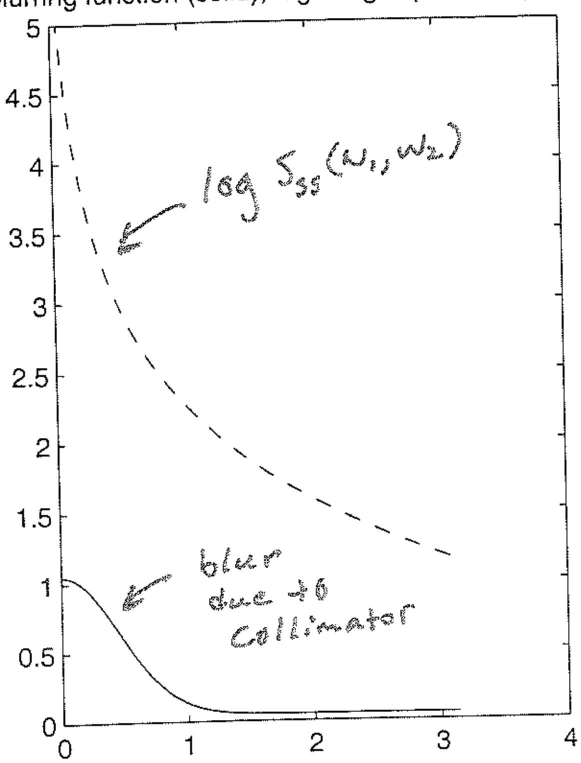
Nuclear Medicine
Spine image

Wiener filter restoration



Wiener filter
restoration

blurring function (solid), log image spectrum (dashed)



Wiener filter (solid), inverse filter (dashed)

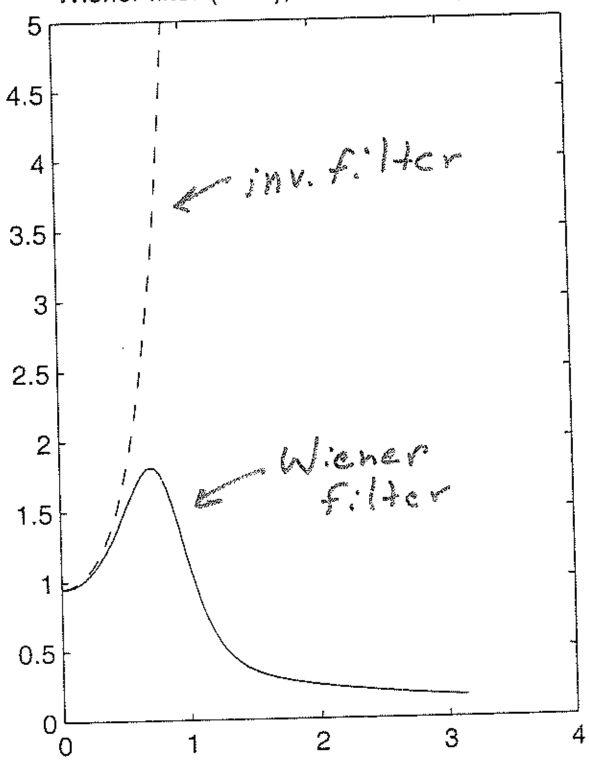


Image Power Spectra

What is a good model for $S_{ss}(\omega_1, \omega_2)$?

Reasonable / Desirable Properties:

Images consist of "smooth" regions and patches seperated by boundaries / edges

"smooth" $\Rightarrow$ low frequency

edges $\Rightarrow$ high frequency

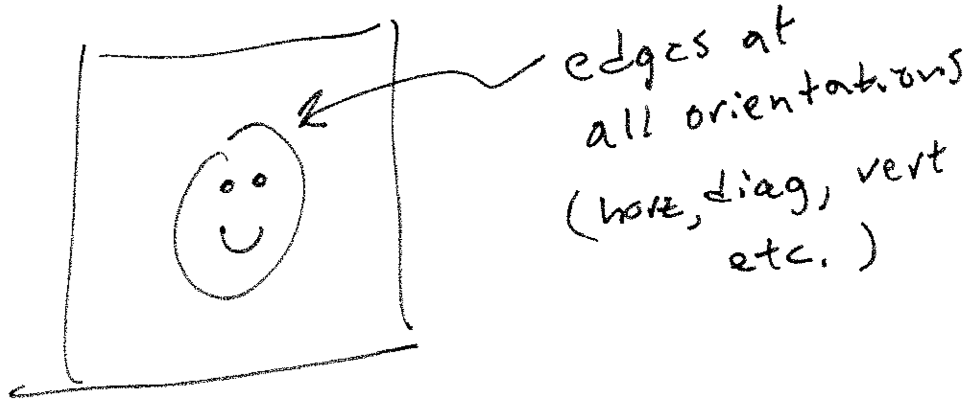
How many pixels are "edge pixels" and how many are "smooth patch pixels"?

Images are mostly "smooth patch pixels" (e.g., 90% of pixels). The rest are edge/texture pixels (e.g., 10%)

⟹ images have more energy at low freqs than at high freqs

Image edges, boundaries, textures occur at all possible orientations.

Ex.



⟹ freq characteristics (energy) should be fairly symmetric in the radial (angular) direction

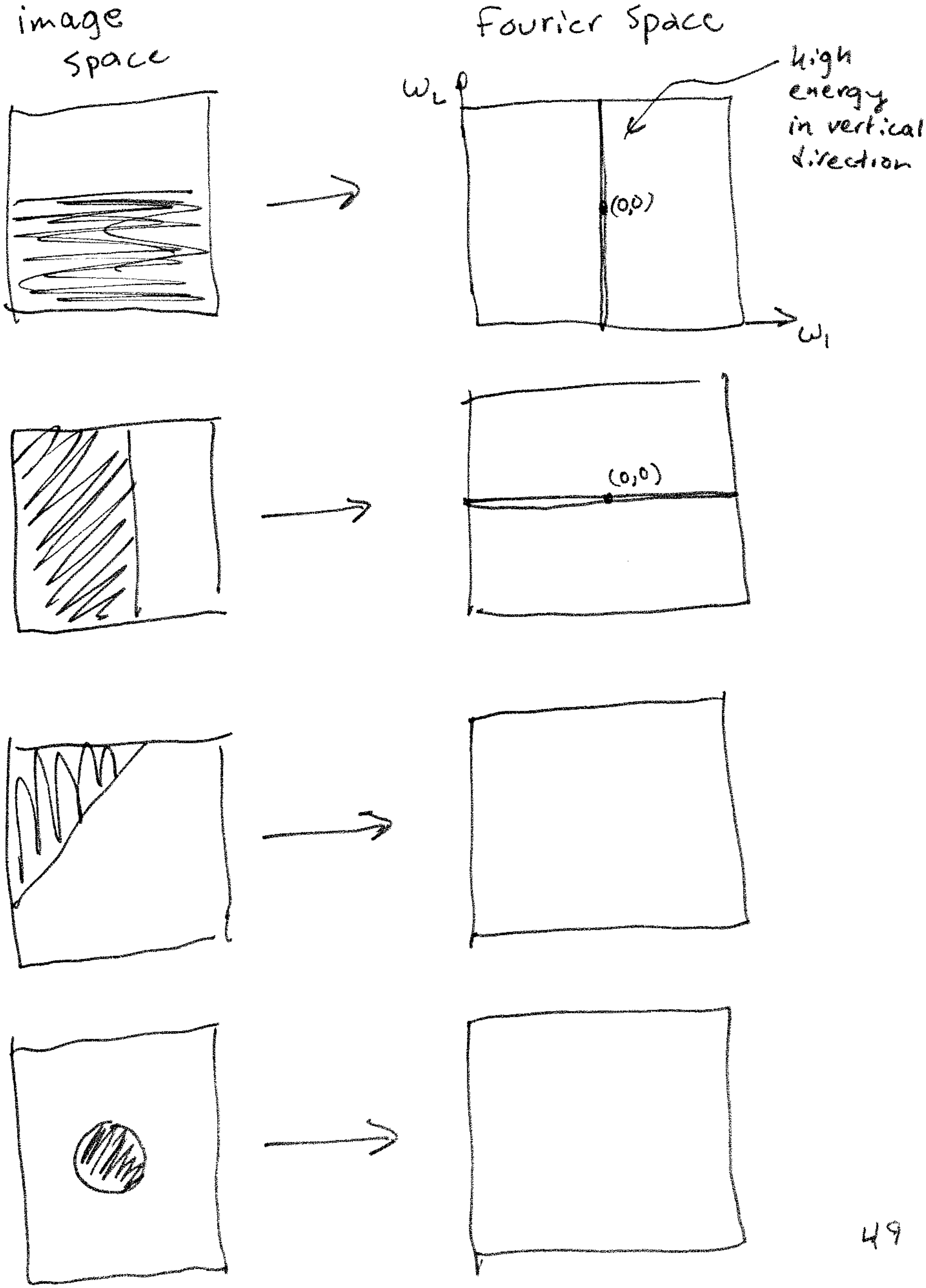


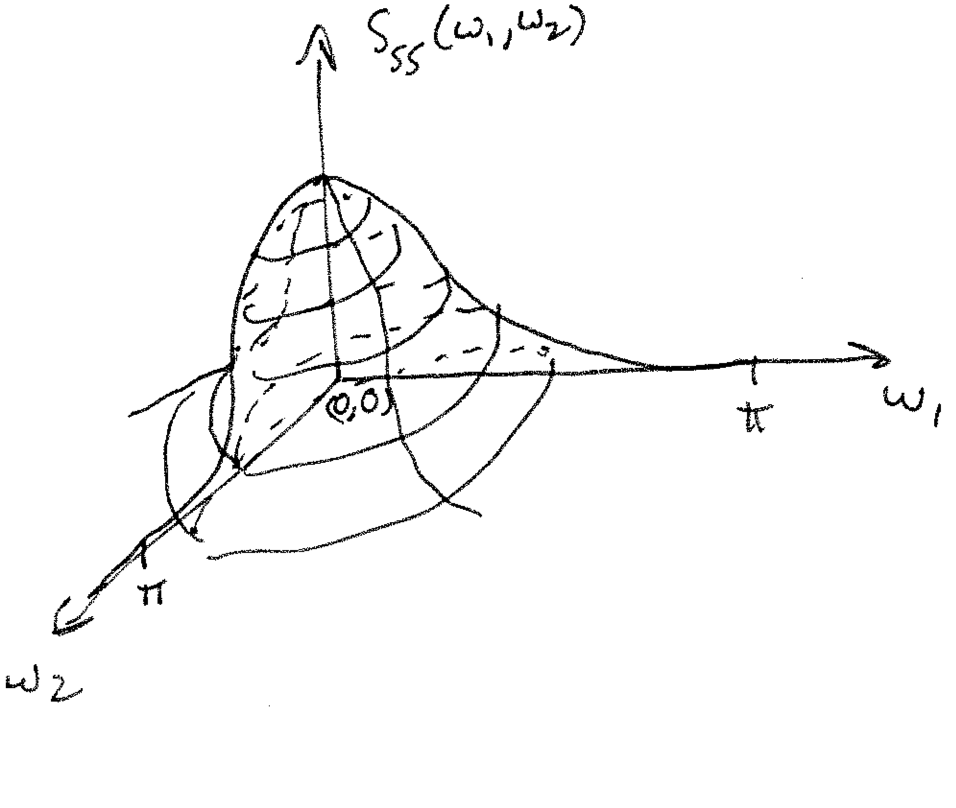
image space
Fourier space
high energy in vertical direction
ω_2
ω_1
(0,0)
(0,0)

Basic model for $S_{ss}(\omega_1, \omega_2)$:

(i)

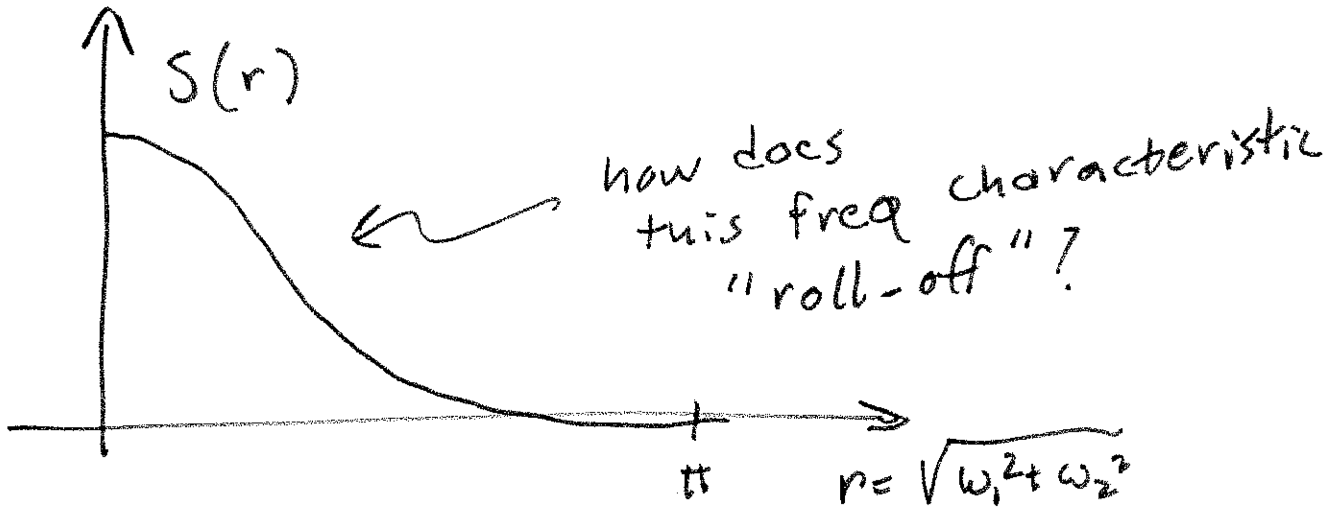
(ii) $S_{ss}(\omega_1, \omega_2)$ depends only on

$r = \sqrt{\omega_1^2 + \omega_2^2}$ (radial symmetry)



Modelling the radial "profile"

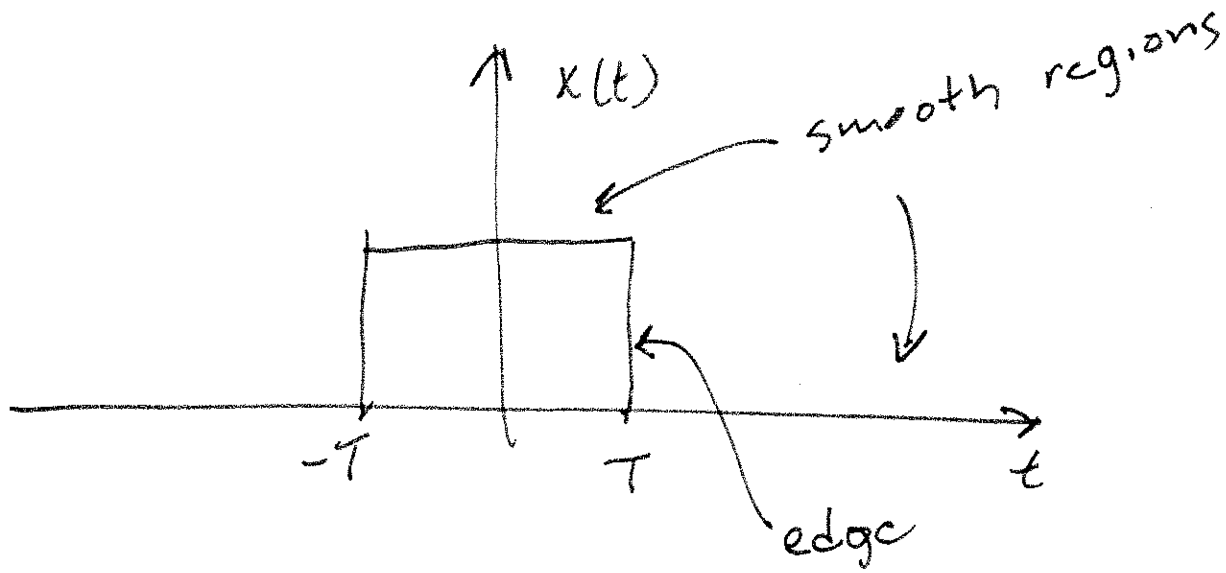
$$S_{ss}(\omega_1, \omega_2) \equiv S_{ss}(r, \phi) = S(r)$$



Think about simple 1-D function with the basic image "structure"

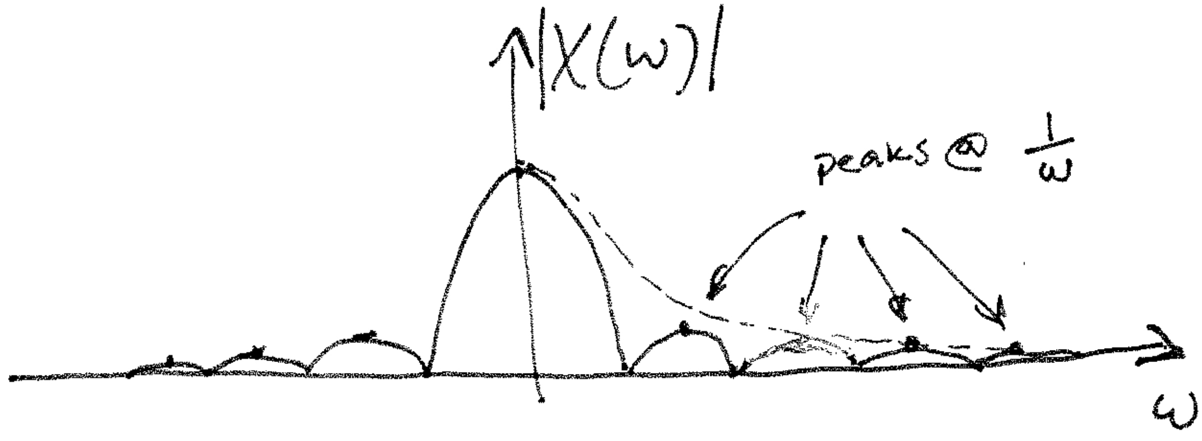
i.e., smooth regions separated by edges

A 1-D "image":



CTFT:

$$X(\omega) = \frac{\sin \omega T}{\omega} \leftarrow \text{sinc}$$



$$|X(\omega)| \leq \frac{1}{|\omega|}$$

The CTFT decays like

$$|X(\omega)| \sim \frac{1}{|\omega|}$$

so the energy / power decays like

$$|X(\omega)|^2 \quad \frac{1}{|\omega|^2}$$

1-D "images" are random superpositions of blocks like this producing a power spectrum

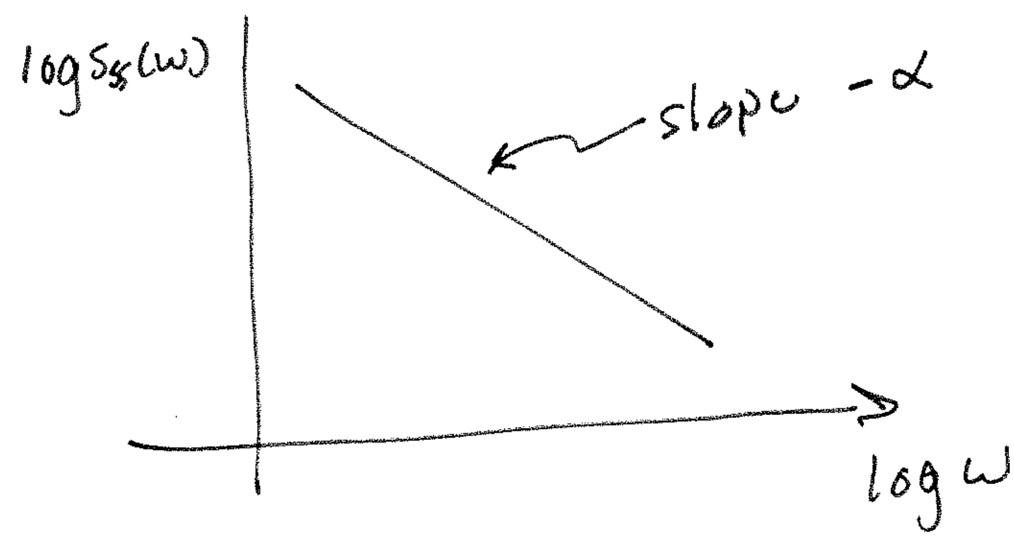
$$S_{ss}(\omega) \propto \frac{1}{|\omega|^2}$$

This called a "1 over frequency" or $1/f$ power spectrum.

In general, a $1/f$ spectrum has the form

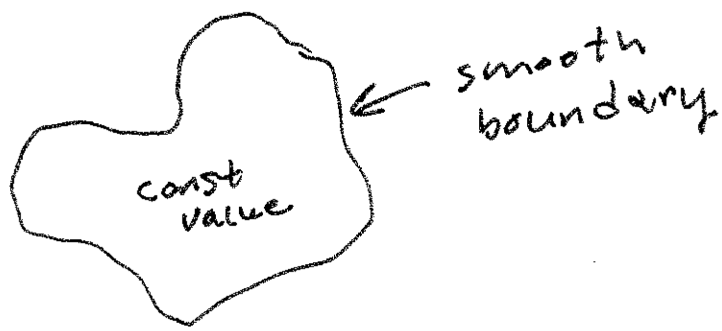
$$S_{SS}(\omega) = \frac{\text{Const.}}{|\omega|^{\alpha}}, \quad \alpha > 0.$$

Notice



So given a power spectrum it is easy to determine the correct α.

We can think of images (2-D) as being random superpositions of 2-D shapes with smooth boundaries and smooth surfaces



This leads to symmetric (radially) 2-D power spectra with $1/f$ characteristics. For real, natural images (collected by cameras) experiments have shown that images behave like

$$S_{ss}(r) \propto \frac{1}{|r|^{\alpha}}$$

$$1 \leq \alpha \leq 3$$

In summary, a reasonable model for image power spectra if

$$S_{ss}(\omega_1, \omega_2) = \frac{\sigma^2}{\left(\sqrt{\omega_1^2 + \omega_2^2 + 1}\right)^{\alpha}}$$

Where σ^2 is the DC power and α is the exponent of the $1/f$ decay.

With this model, we can design a 3-parameter Wiener restoration filter depending on α, σ^2, and σ_w^2 (the white noise power).