Wiener Filtering via the FFT

We can use fast FFT-based convolution to speed-up the Wiener filtering process.

Recall: Wiener filter satisfies the Wiener-Hopf equation

$$\sum_k h[k] \left(R_{ss}[m-k] + R_{ww}[m-k] \right) = R_{ss}[m]$$

$$\Big\updownarrow \text{ DTFT}$$

$$H(\omega) \left(S_{ss}(\omega) + S_{ww}(\omega) \right) = S_{ss}(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{ww}(\omega)}$$

Suppose that we want to Wiener filter an N-point signal

$$x[n], \quad n = 0, \ldots, N-1.$$

To do this, in the DTFT domain we compute

DTFT of $\hat{s}[n]$ — DTFT of $x[n]$

$$\hat{S}(\omega) = H(\omega) \cdot X(\omega)$$

Let's sample this equation in frequency at points $\omega = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$ to get

DFT of $h[n]$ — DFT of $x[n]$ (N-point DFT)

$$\tilde{S}[k] = \underbrace{H[k] \cdot X[k]}_{\text{circular convolution in time}}$$

Taking inverse DFT we obtain

$$\tilde{s}[n] = h[n] \circledast x[n]$$

circular conv.

If $x[n]$ is properly zero-padded,

then $\underbrace{\tilde{s}[n]}_{\substack{\text{true Wiener} \\ \text{filter output via regular conv.}}} = \hat{s}[n]$ on $n = 0, \ldots, N-1$

<u>Ex.</u> Suppose $x[n]$ and $h[n]$ are both N-point sequences.

Then zero-pad both to length $2N-1$ (or nearest power of two $\geq 2N-1$).

<u>Compute FFT of $x[n], h[n]$</u>:

$O(N \log N)$ operations

<u>Multiply</u>:

$\tilde{S}[k] = H[k] \cdot X[k] \longrightarrow O(N)$ ops

<u>Inverse FFT</u>:

$\hat{s}[n]$, $O(N \log N)$ ops

Total: $O(N \log N)$

compare with direct conv. $O(N^2)$ ops

Signal Restoration Using the Wiener Filter

Suppose that we measure a distorted signal

$$s[n] \longrightarrow \underset{\text{LTI}}{\boxed{g[n]}} \longrightarrow x[n]$$

$$x[n] = \sum_k g[k]\, s[n-k]$$

$$X(\omega) = G(\omega) S(\omega)$$

If $|G(\omega)| > 0$ for all $-\pi \le \omega \le \pi$, then we can recover $s[n]$ by computing

$$S(\omega) = \frac{X(\omega)}{G(\omega)}$$

then take inv DTFT of $S(\omega)$.

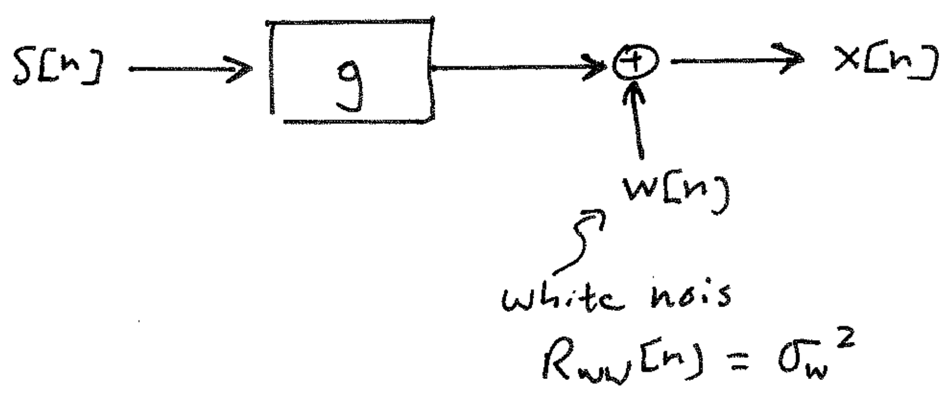
What if $G(\omega) = 0$ for some ω?

The filter $\frac{1}{G(\omega)} = G^{-1}(\omega)$ is called an <u>inverse filter</u>.

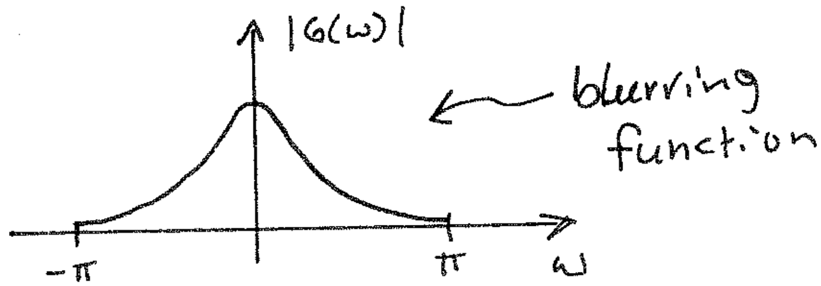
If $G(\omega) = 0$ for some ω, then $G^{-1}(\omega)$ does ot exist.

Even if $G^{-1}(\omega)$ exists, noise in our measurements may severely degrade results.

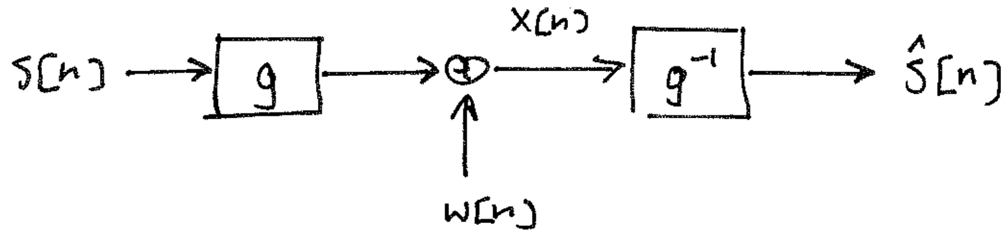
<u>Ex.</u>



Assume



Ex. (Cont.)



In frequency domain,

$$\hat{S}(\omega) = \frac{G(\omega)S(\omega)}{G(\omega)} + \frac{W(\omega)}{G(\omega)}$$

$$= S(\omega) + \frac{W(\omega)}{G(\omega)}$$

IDTFT

$$\Rightarrow \quad \hat{s}[n] = s[n] + \underbrace{g^{-1}[n] * w[n]}$$

where $g^{-1}[n] = \text{IDTFT}\left(\frac{1}{G(\omega)}\right)$

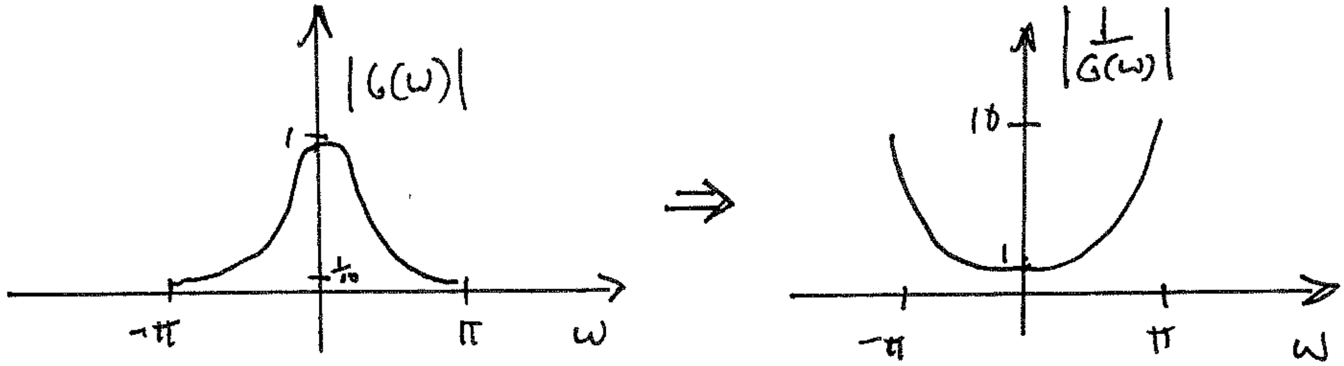
noise at output of filter, $w'[n]$

Power spectral density of output noise:

$$S_{w'w'}(\omega) = \frac{S_{ww}(\omega)}{|G(\omega)|^2} = \frac{\sigma_w^2}{|G(\omega)|^2}$$

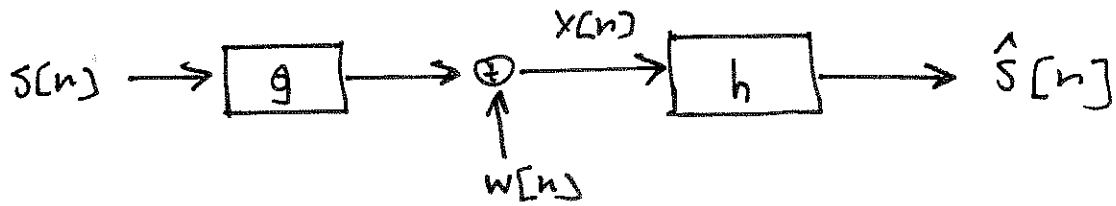
const. for all ω

Recall



The inverse filter drastically amplifies high frequency noise!!

<u>Wiener Filter Solution</u>:



Design $h[n]$ to minimize the MSE $E[(s[n]-\hat{s}[n])^2]$

As always, the optimal MMSE Wiener filter satisfies

$$R_{xs}[m] = \sum_k h[k]\, R_{xx}[m-k]$$

or

$$S_{xs}(\omega) = H(\omega)\, S_{xx}(\omega)$$

$R_{xs}[m] =$

$$E\Big[\, x[n-m]\, s[n] \,\Big] = E\left[\left(\sum_k g[k]\, s[n-m-k] + w[n-m]\right)\cdot [n]\right]$$

$$= E\left[\sum_k g[k]\, s[n-m-k]\, s[n]\right],$$

since $w[n]$ and $s[n]$ are indep. & zero-mean

$$R_{xs}[m] = \sum_k g[k]\, R_{ss}[m+k]$$

note $+k$ not $-k$ $\Rightarrow$ correlation not conv.

$\updownarrow$ DTFT

$$\boxed{S_{xs}(\omega) = G^*(\omega)\, S_{ss}(\omega)}$$

$$R_{xx}[m] = E\Big[\, x[n]\, x[n-m] \,\Big]$$

$$= E\left[\left(\sum_k g[k]\, [n-k] + w[n]\right)\left(\sum_\ell g[\ell]\, s[n-m-\ell] + w[n-m]\right)\right]$$

$$= E\left[\sum_k \sum_\ell g[k]\, g[\ell]\, s[n-k]\, s[n-m-\ell]\right] + E\Big[\, w[n]\, w[n-m] \,\Big]$$

by indep. & zero-mean assumptions

$$= \sum_k \sum_\ell g[k]\, g[\ell]\, R_{ss}[m+\ell-k] + R_{ww}[m]$$

$$R_{xx}[m] = \sum_k \sum_l g[k]\, g[l]\, R_{ss}[m+l-k] + R_{ww}[m]$$

$$\Big\updownarrow \text{ DTFT}$$

$$S_{xx}(\omega) = G^*(\omega)\, G(\omega)\, S_{ss}(\omega) + S_{ww}(\omega)$$

$$\boxed{S_{xx}(\omega) = |G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega)}$$

Thus,

$$S_{xs}(\omega) = H(\omega)\, S_{sx}(\omega)$$

$$\Rightarrow \quad G^*(\omega)\, S_{ss}(\omega) = H(\omega) \left(|G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega) \right)$$

$$\Rightarrow \quad \boxed{H(\omega) = \frac{G^*(\omega)\, S_{ss}(\omega)}{|G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega)}}$$

Optimal Wiener "restoration" filter

Comparison with simple inv. filter:

If $S_{ss}(\omega) \gg S_{ww}(\omega)$, then

$$H(\omega) \approx \frac{1}{G(\omega)}$$

If $S_{ss}(\omega) \ll S_{ww}(\omega)$, then

$$H(\omega) \approx 0.$$

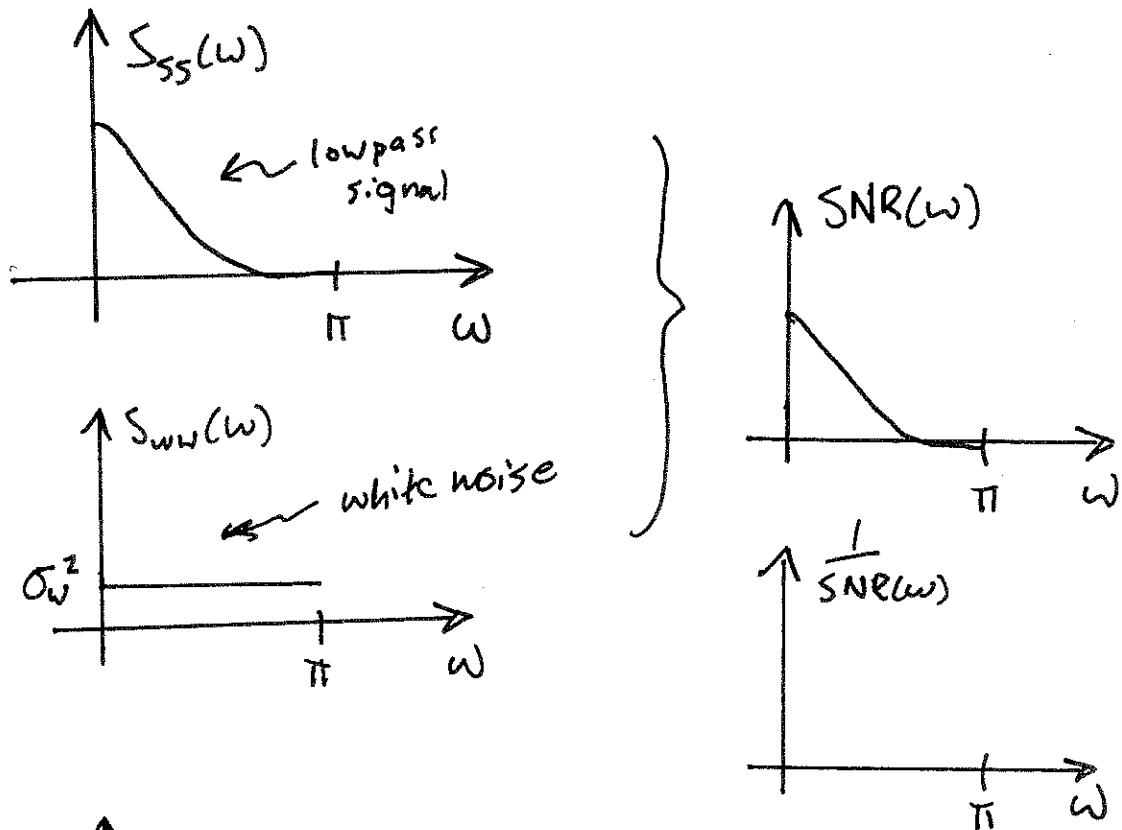
Note:

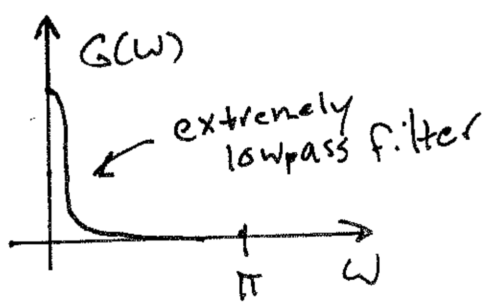
$$H(\omega) = \frac{G^*(\omega)}{|G(\omega)|^2 + \frac{1}{SNR(\omega)}}$$

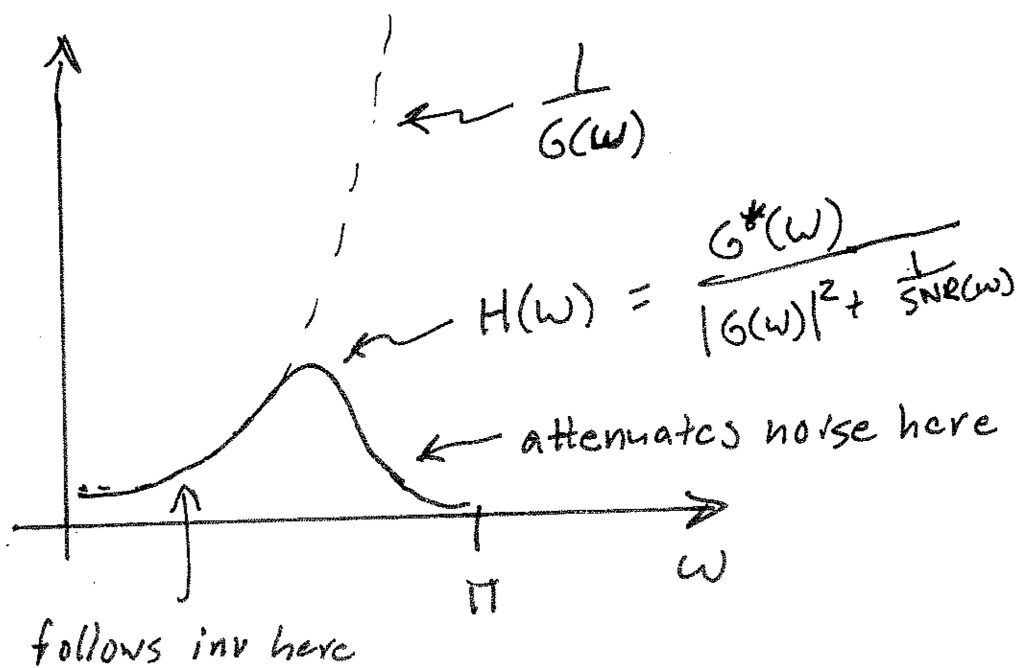
where $SNR(\omega) \equiv \dfrac{S_{ss}(\omega)}{S_{ww}(\omega)}$

\# as long as $SNR(\omega) < \infty$ (we have some noise)
$H(\omega)$ is well-defined !!

Ex.


$S_{ss}(\omega)$
lowpass signal
π
ω
$S_{ww}(\omega)$
white noise
σ_w^2
π
ω
$SNR(\omega)$
π
ω
$\frac{1}{SNR(\omega)}$
π
ω


$G(\omega)$
extremely lowpass filter
π
ω


$\frac{1}{G(\omega)}$
$H(\omega) = \frac{G^*(\omega)}{|G(\omega)|^2 + \frac{1}{SNR(\omega)}}$
attenuates noise here
π
ω
follows inv here