Daubechies' Wavelets

Theorem

The DTFT of $h(n)$ that has $K \leq \frac{N}{2}$ zeros at $\omega = \pi$:

$$H(\omega) = \left(\frac{1 + e^{i\omega}}{2}\right)^K \mathcal{L}(\omega),$$

satisfies

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$

if and only if $L(\omega) = |\mathcal{L}(\omega)|^2$ can be written as

$$L(\omega) = P\left(\sin^2\left(\frac{\omega}{2}\right)\right)$$

where $P(\cdot)$ is a polynomial of the form

$$P(y) = \sum_{k=0}^{K-1} \binom{K-1+k}{k} y^k + y^K R\left(\tfrac{1}{2} - y\right)$$

and $R(\cdot)$ is an odd polynomial chosen so that $P(y) \geq 0$ for $0 \leq y \leq 1$.

This theorem shows that a K-regular scaling filter (i.e., a scaling filter associated with a wavelet with K vanishing moments) satisfies

$$|H(\omega)|^2 = \left|\frac{1+e^{i\omega}}{2}\right|^{2K} P\left(\sin^2\left(\frac{\omega}{2}\right)\right)$$

To find $H(\omega)$, and hence $h(n)$, we must find the square root of this equation. This is known as <u>spectral factorization</u>. ← carried out numerically, in general

It turns out that several factorizations will work. That is, this does not give a unique $h(n)$. The Daubechies' filters we looked at earlier are the solutions with <u>minimum phase</u>.

Ex. D4 scaling filter

$h(0) = (1 - \cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3}))/(2\sqrt{2}) = 0.48296$

$h(1) = (1 + \cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3}))/(2\sqrt{2}) = 0.83652$

$h(2) = (1 + \cos(\frac{\pi}{3}) - \sin(\frac{\pi}{3}))/(2\sqrt{2}) = 0.22414$

$h(3) = (1 - \cos(\frac{\pi}{3}) - \sin(\frac{\pi}{3}))/(2\sqrt{2}) = -0.12941$

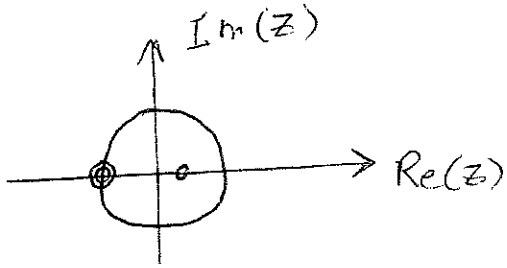
Vanishing moments $K = N/2 = 2$

k	$\mu_1(k)$	$m_1(k)$
0	0	0
1	0	0
2	1.2247	0.2165
3	6.5720	0.7868
	wavelet filter $h_1(n)$	wavelet fnc $\psi(t)$

} 1st two moments are zero $\Rightarrow$ wavelets are "blind" to linear functions

zeros of $H(z) = \sum_{n=0}^{3} h(n) z^{-n}$

$z = -1, -1, 0.2679$



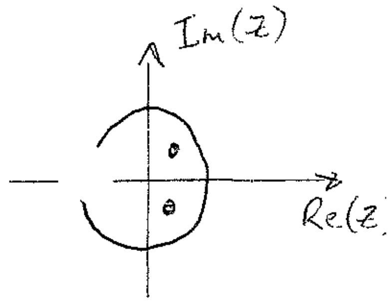
Ex. D6 scaling filter

n	$h(n)$	$h_1(n) = (-1)^{n+1} h(5-n)$
0	0.3327	-0.0352
1	0.8069	-0.0854
2	0.4599	0.1350
3	-0.1350	0.4599
4	-0.0854	-0.8069
5	0.0352	0.3327

Vanishing Moments $K = \frac{N}{2} = 3$

K	$\mu_1(k)$	
0	0	0
1	0	0
2	0	0
3	3.3541	0.2965
4	40.6797	2.2825
5	329.3237	11.4461
	wavelet filter $h_1(n)$	wavelet fnc $\psi(t)$

1st three moments are zero
$\Rightarrow$ wavelets are "blind" to quadratic polynomials

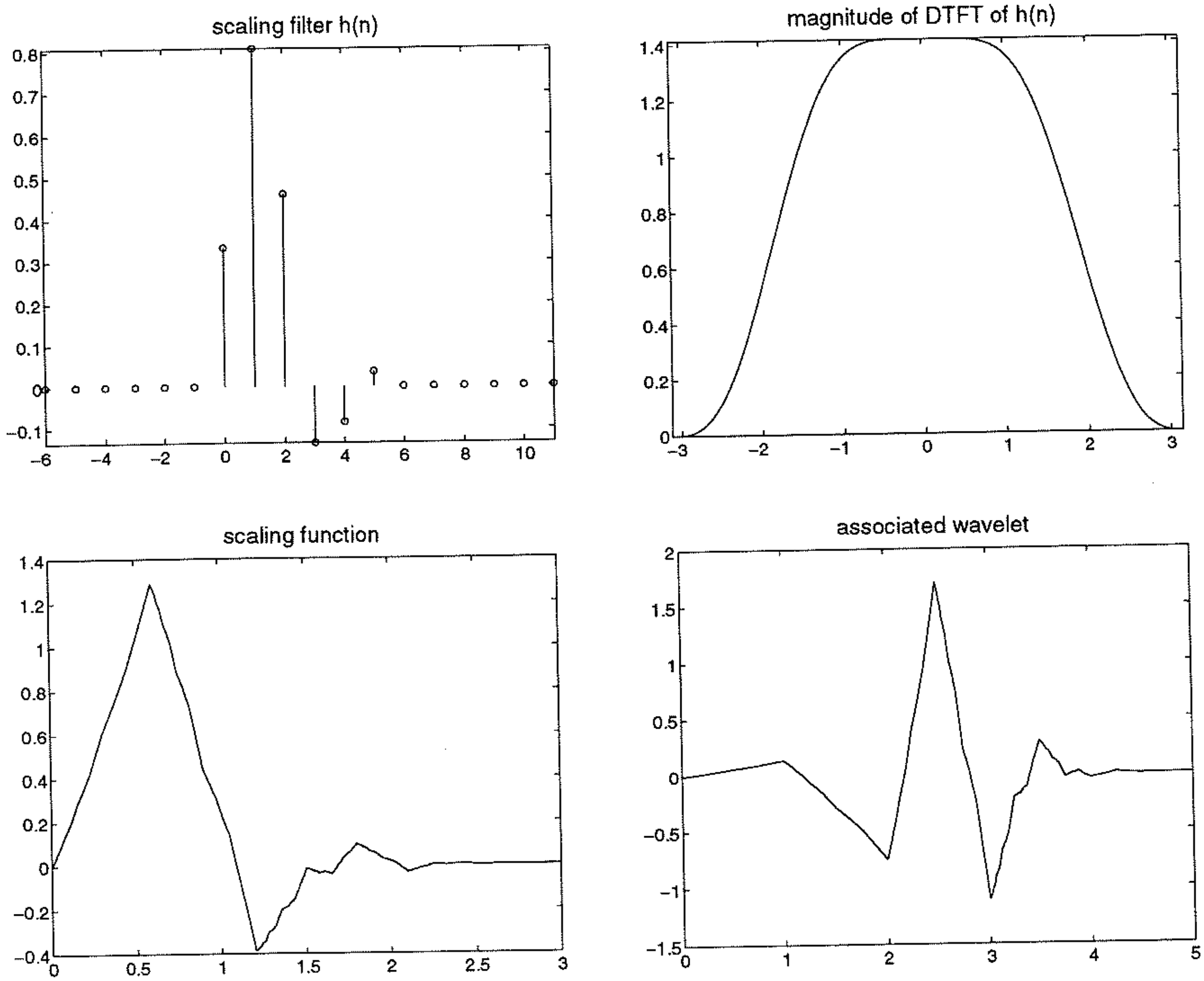


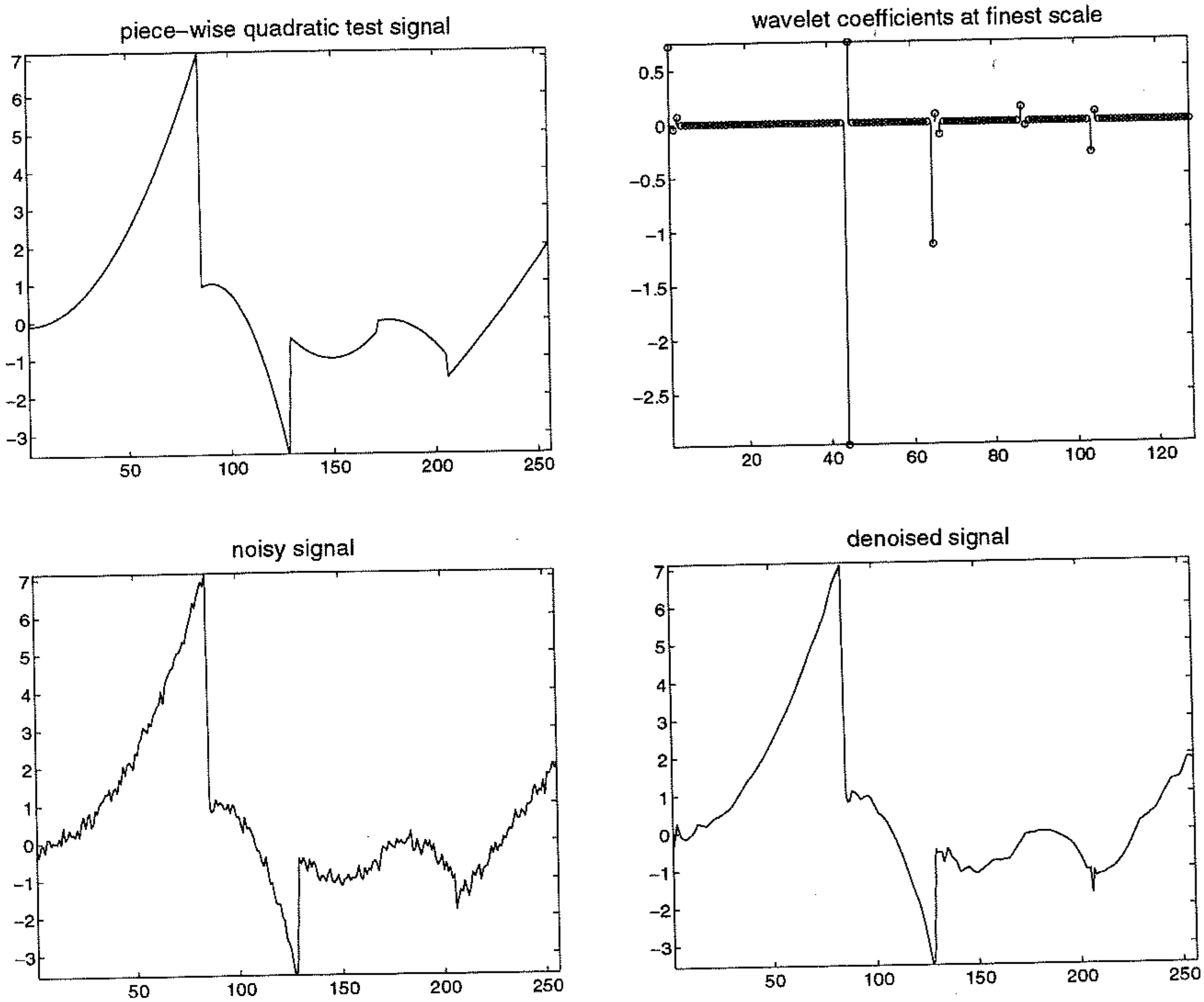
Zeros of $H(z)$

$z = -1, -1, -1, \; 0.2873 \pm 0.1529i$

$\alpha = 1.3598\ldots$

$\beta = -0.7821\ldots$




piece–wise quadratic test signal
7
6
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4
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100
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wavelet coefficients at finest scale
0.5
0
−0.5
−1
−1.5
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20
40
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80
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noisy signal
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denoised signal
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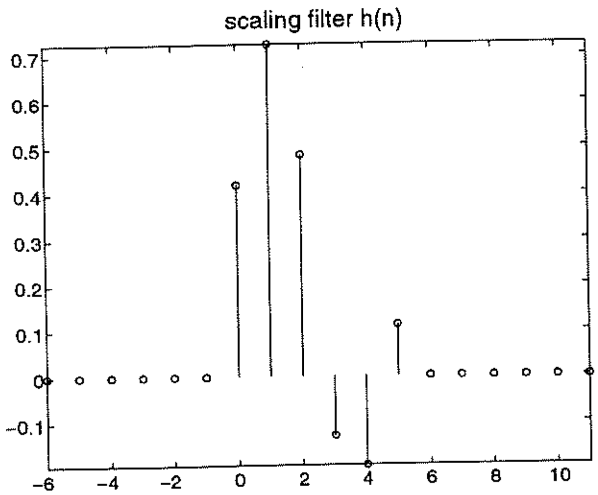
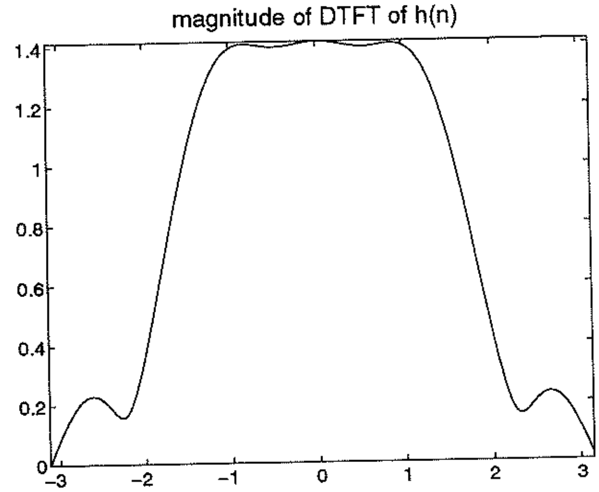
moments

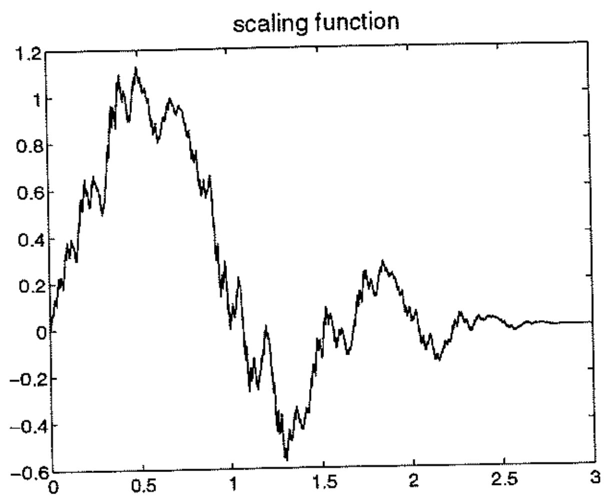
k	$\mu_1(k)$
0	0
1	1.0860
2	1.1876
3	2.6798
4	18.3214
5	136.1845

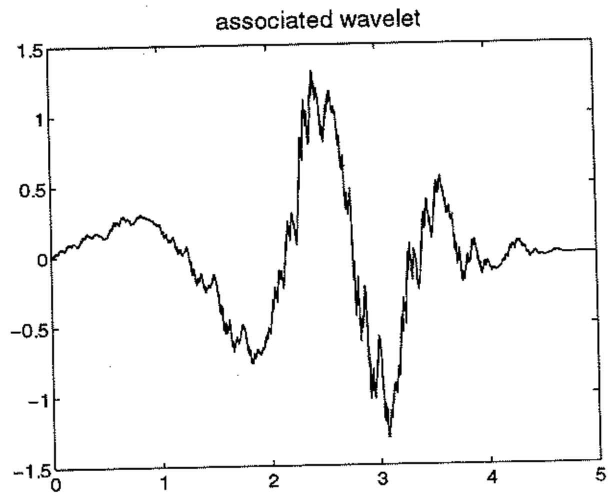
$\alpha = \frac{\pi}{3}$

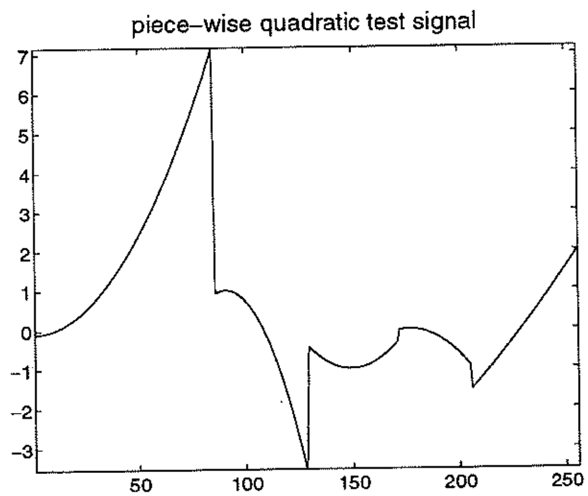
$\beta = -\frac{\pi}{3}$







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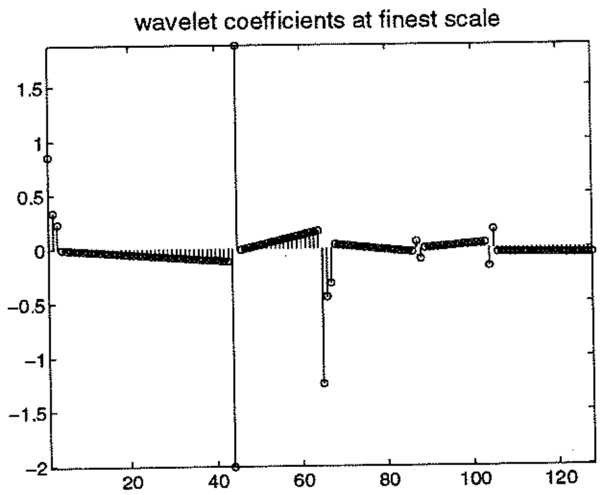
wavelet coefficients at finest scale
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100
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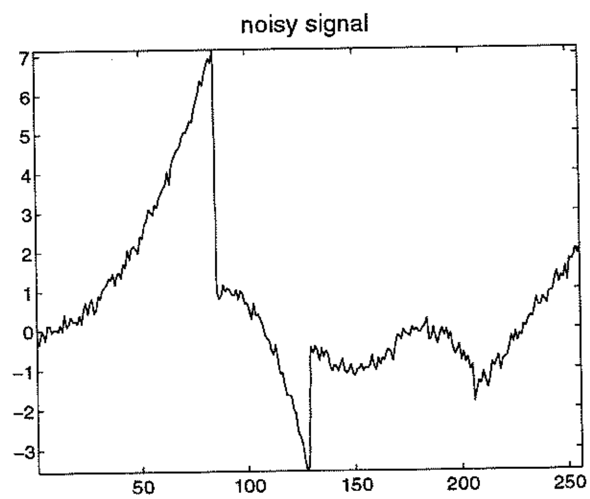
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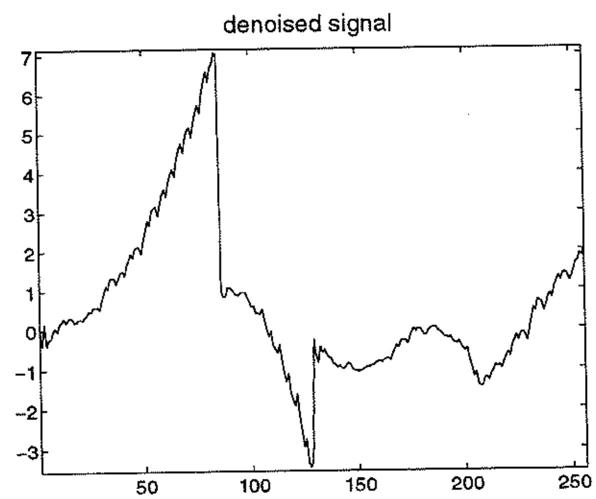
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Daubechies' Method For Wavelet Design

We seek a scaling filter $h(n)$ satisfying

1. $\sum_n h(n) = \sqrt{2}$

2. $\sum_n h(n)\, h(n-2k) = \delta(k)$

3. $\sum_n n^k\, h_1(n) = 0 \quad , \quad k = 0, 1, \ldots, K-1$

   or equivalently

   $$\sum_n n^k\, (-1)^n\, h(1-n) = 0$$

Note 1. and 3. are linear equations in $h(n)$, 2. is quadratic.

Also, in order to have K vanishing moments, $h(n)$ must be of length $2K$ or more.

Assuming $h(n)$ is length $N = 2K$, we have precisely K linear equations, K quadratic equations, and $2K$ unknowns. How do we solve this system of <u>nonlinear</u> equations?

- Recall that 3. (vanishing moments) is equivalent to K regularity of $h(n)$:

$$H(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^{K} \mathcal{L}(\omega)$$

This implies that

$$M(\omega) \equiv |H(\omega)|^2 = \left|\frac{1+e^{i\omega}}{2}\right|^{2K} L(\omega)$$

where

$$L(\omega) \equiv |\mathcal{L}(\omega)|^2$$

- Note

$$\left|\frac{1+e^{i\omega}}{2}\right|^2 = \left(\frac{1+e^{i\omega}}{2}\right)\left(\frac{1+e^{-i\omega}}{2}\right) = \frac{1+\cos(\omega)}{2}$$

$$= \cos^2\left(\frac{\omega}{2}\right)$$

So

$$M(\omega) = \left|\cos^2\left(\frac{\omega}{2}\right)\right|^{K} L(\omega)$$

Also, $|H(\omega)|^2 = H(\omega)\,H^*(\omega) = H(\omega)\,H(-\omega)$
(since $h(n)$ is real-valued)

$\Rightarrow |H(\omega)|^2 = |H(-\omega)|^2$

$\Rightarrow |H(\omega)|^2$ is even fnc of ω.

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We have

$$\underbrace{M(\omega)}_{\text{even}} = \underbrace{\left|\cos^2\left(\tfrac{\omega}{2}\right)\right|^K}_{\text{even}} \; \underbrace{L(\omega)}_{\text{even}}$$

So $L(\omega)$ is a polynomial in $\cos(\omega)$
(instead of $e^{i\omega} = \cos(\omega) + i\sin(\omega)$)

It is convenient to express $L(\omega)$
in terms of $\sin^2\left(\frac{\omega}{2}\right) = (1-\cos(\omega))/2$ instead:

$$L(\omega) \doteq P\left(\sin^2\left(\tfrac{\omega}{2}\right)\right) = \overset{\text{degree } K-1}{\text{polynomial in } \sin^2\left(\tfrac{\omega}{2}\right)}$$

Now we have

$$M(\omega) = \left|\cos^2\left(\tfrac{\omega}{2}\right)\right|^K P\left(\sin^2\left(\tfrac{\omega}{2}\right)\right)$$

Recall that Q. (quadratic constraints)

$$\Longrightarrow \quad M(\omega) + M(\omega + \pi) = 1$$ (see p. 148)

In terms of P, this becomes

$$\left|\cos^2\left(\tfrac{\omega}{2}\right)\right|^K P\left(\sin^2\left(\tfrac{\omega}{2}\right)\right) + \left|\sin^2\left(\tfrac{\omega}{2}\right)\right|^K P\left(\cos^2\left(\tfrac{\omega}{2}\right)\right) =$$

Let $y \equiv \cos^2(\frac{\omega}{2})$. Then we have

$$y^K P(1-y) + (1-y)^K P(y) = 1$$

for all $y \in [0,1]$. Since P is a polynomial, this implies that we have equality for all $y \in \mathbb{R}$.

To find P satisfying this equation we need Bezout's theorem.

<u>Thm:</u> If p_1 and p_2 are polynomials of degree n_1 and n_2, respectively, with no common zeros, then there exist unique polynomials q_1 and q_2 of degree n_2-1 and n_1-1, respectively, such that

$$p_1(x)q_1(x) + p_2(x)q_2(x) = 1$$

<u>proof</u>: straightforward inductive argument.

Bezout's theorem shows that

$$y^K p(1-y) + (1-y)^K p(y) = 1$$

has a unique solution $p(y)$, which is a polynomial of degree $\leq K-1$.

Re-arranging this expression we have

$$p(y) = \left(1 - y^K p(1-y)\right)(1-y)^{-K}$$

The Taylor's series of $(1-y)^{-K}$ is

$$(1-y)^{-K} = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k + R_K(y)$$

where

$$R_K(y) = \sum_{k \geq K} a_k y^k$$

So

$$P(y) = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k + R_K(y) \quad \underbrace{R_K(y)\, y^K p(1-y)}_{= \sum_{k \geq K} a_k' y^k}$$

We know that degree $P(y) \leq K-1$,

hence

$$P(y) = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k$$

Thus, we now have

$$|H(\omega)|^2 = |\cos^2(\tfrac{\omega}{2})|^K P(\sin^2(\tfrac{\omega}{2}))$$

$$= \left| \left(\frac{1+e^{j\omega}}{2}\right)\left(\frac{1+e^{-j\omega}}{2}\right) \right|^K P\left(\frac{1}{2} - \frac{e^{j\omega}}{2} - \frac{e^{-j\omega}}{2}\right)$$

or with $z = e^{j\omega}$

$$|H(z)|^2 = \left| \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \right|^K P\left(\frac{1}{2} - \frac{z}{2} - \frac{z^{-1}}{2}\right)$$

$|H(z)|^2$ is simply a polynomial in z. We can factor it using the "roots" command in Matlab.

If we choose the $N-1$ roots of smallest magnitude, then we get $N-1$ roots corresponding to a minimum phase factorization of $H(z)$. If $z_0, \ldots, z_{N-1}$ are these roots, then

$$H(z) = \prod_{n=0}^{N-1} (z^{-1} - z_n)$$

$$= \sum_{n=0}^{N-1} h(n) z^{-n}$$

and hence we can identify $h(n)$. This is the standard Daubechies length N filter.