

Wavelet System Design

We have established two key conditions on the scaling filter that are necessary to insure a perfect reconstruction filterbank and a valid, orthogonal MRA:

$$(i) \sum_n h(n) = \sqrt{2}$$

$$(ii) \sum_n h(n)h(n-k) = \delta(k)$$

These conditions are design constraints. The remaining degrees of freedom in $h(n)$ can be used to obtain different wavelet transforms with various desirable properties.

In particular, we will design $h(n)$ so that the wavelet basis function have a certain number of "Vanishing moments". This property is critical for signal and image compression, approximation, and denoising.

Before getting into design, let's first briefly look at how one goes about determining the continuous scaling and wavelet functions associated with a given scaling filter.

Calculating the Scaling Function

As mentioned earlier, most scaling functions and wavelets do not have closed-form expressions.

So, what do they look like and how can we calculate them?

Method of Successive Approximation:

We know that $\phi(t)$ satisfies

$$\phi(t) = \sum_{n=0}^{N-1} h(n) \sqrt{2} \phi(2t-n)$$

for a length N scaling filter.

So consider the iteration

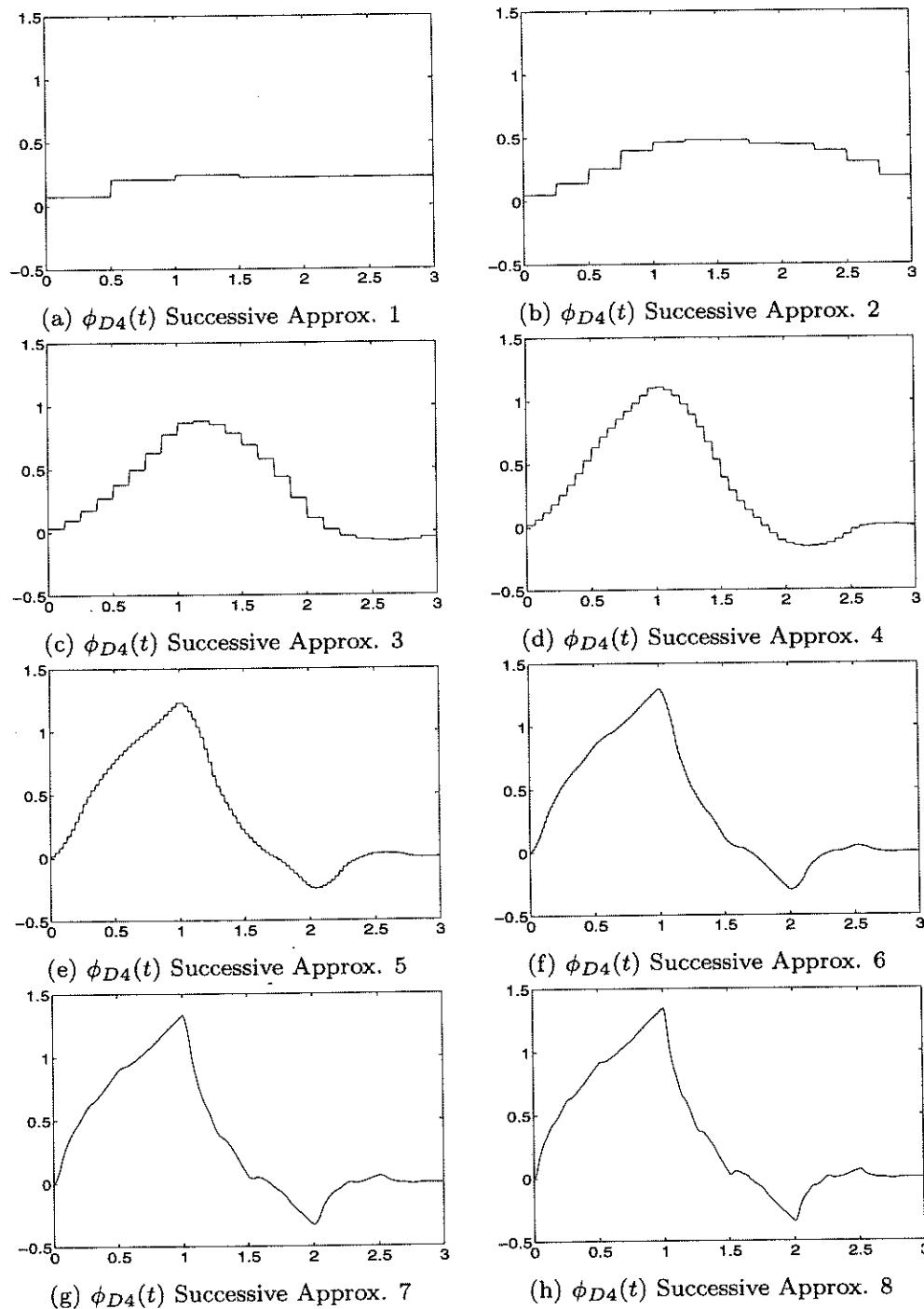
$$\phi^{(k+1)}(t) = \sum_{n=0}^{N-1} h(n) \sqrt{2} \phi^{(k)}(2t-n)$$

$k = 0, 1, \dots$ with some initial $\phi^{(0)}(t)$.

Typically, we usually start the iteration with $\phi^{(0)}(t)$ set to a box function with the proper support (i.e., for a length N scaling filter, the support of $\phi(t)$ is $[0, N-1]$.)

The corresponding wavelet is found by

$$\psi^{(k+1)}(t) = \sum_{n=0}^{N-1} (-1)^{1-n} h(1-n) \phi^{(k)}(2t-n)$$

Figure 5.3. Iterations of the Successive Approximations for φ_{D4}

at $\omega = 8(2k+1)\pi$, etc. Because (5.74) is a product of stretched versions of $H(\omega)$, these zeros of $H(\omega/2^j)$ are the zeros of the Fourier transform of $\varphi(t)$. Recall from Theorem 15 that $H(\omega)$ has no zeros in $-\pi/3 < \omega < \pi/3$. All of this gives a picture of the shape of $\Phi(\omega)$ and the location of its zeros. From an asymptotic analysis of $\Phi(\omega)$ as $\omega \rightarrow \infty$ one can study the smoothness of $\varphi(t)$.

Refinement Matrix Method

Another way to determine the scaling function is as follows.

Let $\underline{\phi} = \begin{bmatrix} \phi(0) \\ \phi(1) \\ \vdots \\ \phi(N-1) \end{bmatrix}$

Then

$$\begin{bmatrix} h(0) & 0 & \cdots & \cdots & 0 \\ h(2) & h(1) & h(0) & \cdots & 0 \\ \vdots & & & & \\ h(N-2) & h(N-3) & \cdots & h(0) & \cdots 0 \\ 0 & h(N-1) & h(N-2) & \cdots & h(1) \\ 0 & 0 & h(N-1) & \cdots & h(3) \\ 0 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & h(N-1) \end{bmatrix} \underline{\phi} = \underline{\phi}$$

M = "refinement matrix"

This relationship is a consequence of the fact that $\phi(t)$ has compact support and $\phi(t) = \sum_{n=0}^{N-1} h(n) \sqrt{2} \phi(2t-n)$

Ex. $N=4$ $h(n) = \{ \dots, h(0), h(1), h(2), h(3), 0, \dots \}$

$$\phi(t) \text{ support} = [0, 3]$$

$$\phi(t) = \sum_{n=0}^3 h(n) \sqrt{2} \phi(2t-n)$$

$$\Rightarrow \phi(0) = h(0) \sqrt{2} \phi(0)$$

$$\begin{aligned} \phi(1) &= h(0) \sqrt{2} \phi(2) + h(1) \sqrt{2} \phi(1) \\ &\quad + h(2) \sqrt{2} \phi(0) \end{aligned}$$

$$\begin{aligned} \phi(2) &= h(0) \sqrt{2} \phi(4)^0 + h(1) \sqrt{2} \phi(3)^0 \\ &\quad + h(2) \sqrt{2} \phi(2) + h(3) \sqrt{2} \phi(1) \end{aligned}$$

$$\begin{aligned} \phi(3) &= h(0) \sqrt{2} \phi(6)^0 + h(1) \sqrt{2} \phi(5)^0 \\ &\quad + h(2) \sqrt{2} \phi(4)^0 + h(3) \sqrt{3} \phi(3) \end{aligned}$$

or

$$\sqrt{2} \begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(2) & h(1) & h(0) & 0 \\ 0 & h(3) & h(2) & h(1) \\ 0 & 0 & 0 & h(3) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{M_0}$

$\Rightarrow \underline{\phi}$ is eigenvector of M_0
eigenvalue = 1

Thus, we see that evaluation of $\phi(t)$ on the integers boils down to solving the "eigen system"

$$M \underline{\phi} = \underline{\phi}$$

Once we have $\underline{\phi}$, we can find evaluations of $\phi(t)$ at the half-integers according to the basic scaling equation:

$$\phi\left(\frac{n}{2}\right) = \sum_{k=0}^{N-1} h(k) \sqrt{2} \phi(n-k)$$

Repeating this process, we can find evaluations of $\phi(t)$ (and $\psi(t)$) on the dyadic rationals

$$t = \frac{n}{2^m}, \quad m, n \in \mathbb{Z}.$$

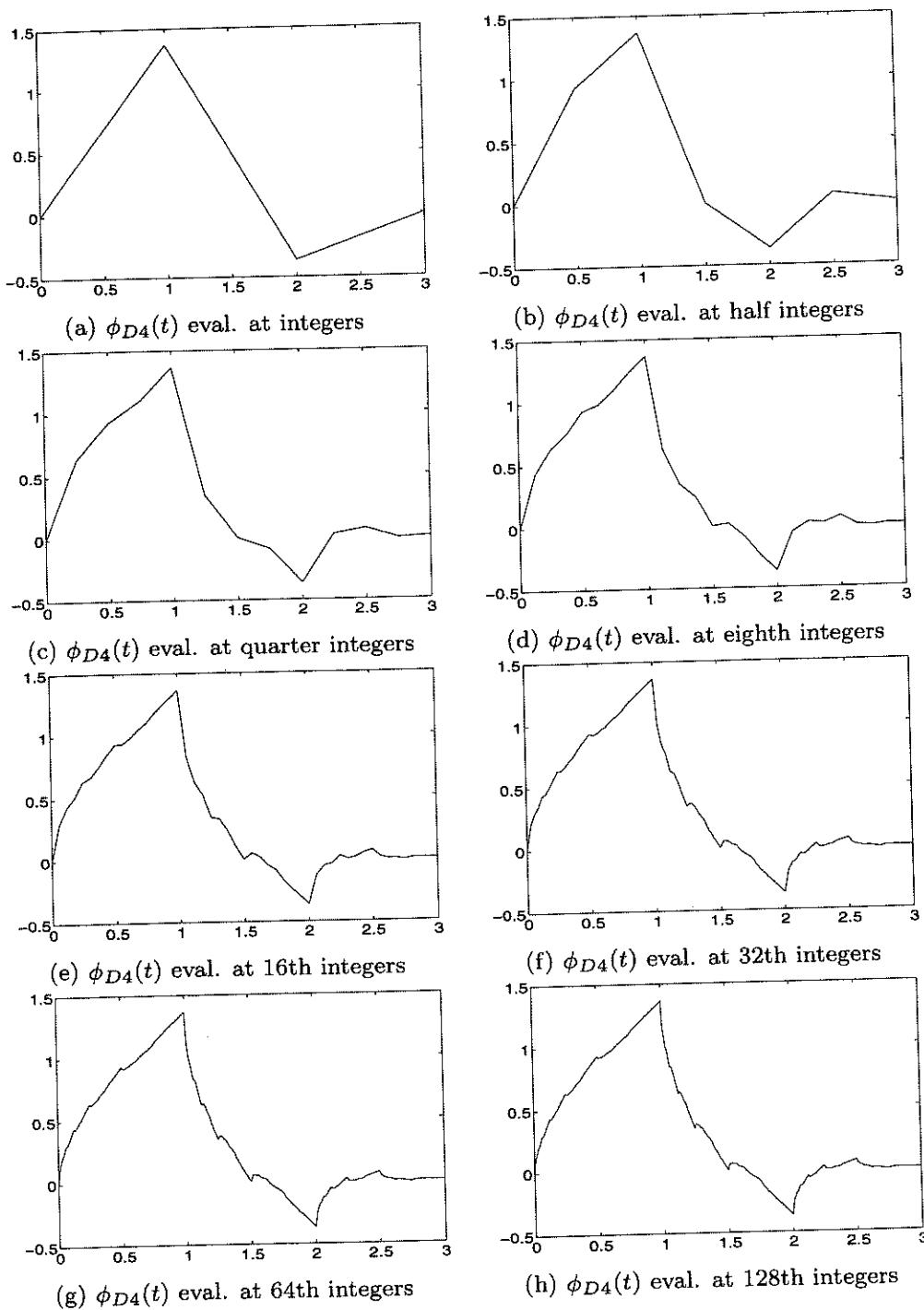


Figure 5.5. Iterations of the Dyadic Expansion for ϕ_{D4}

Not only does this dyadic expansion give an explicit method for finding the exact values of $\varphi(t)$ of the dyadic rationals ($t = r/2^j$), but it shows how the eigenvalues of M say something about the $\varphi(t)$. Clearly, if $\varphi(t)$ is continuous, it says everything.

MATLAB programs are included in Appendix C to implement the successive approximation and dyadic expansion approaches to evaluating the scaling function from the scaling coefficients. They were used to generate the figures in this section. It is very illuminating to experiment with different $h(n)$ and observe the effects on $\varphi(t)$ and $\psi(t)$.

FIR Scaling Filter Design

Let h be a length N scaling filter. Recall that h must satisfy

$$(\text{linear}) \quad \sum_{n=0}^{N-1} h(n) = \sqrt{2}$$

$$(\text{quadratic}) \quad \sum_{n=0}^{N-1} h(n)h(n-2k) = \delta(k)$$

Length-2

$$\left. \begin{array}{l} h(0) + h(1) = \sqrt{2} \\ h^2(0) + h^2(1) = 1 \end{array} \right\} \begin{matrix} 2 \text{ degrees of} \\ \text{ freedom, 2} \\ \text{ constraints!} \end{matrix}$$

$$\Rightarrow \{h(0), h(1)\} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

the Haar scaling function is the only length-2 filter satisfying the linear and quadratic constraints!

Length - 4

linear: $h(0) + h(1) + h(2) + h(3) = \sqrt{2}$

quadratic: $h^2(0) + h^2(1) + h^2(2) + h^2(3) = 1$

$$h(0)h(2) + h(1)h(3) = 0$$

Three equations, four unknowns

\Rightarrow 1 extra degree of freedom

It turns out that solutions to the three equations above take the general form

$$h(0) = (1 - \cos \alpha + \sin \alpha) / (2\sqrt{2})$$

$$h(1) = (1 + \cos \alpha + \sin \alpha) / (2\sqrt{2})$$

$$h(2) = (1 + \cos \alpha - \sin \alpha) / (2\sqrt{2})$$

$$h(3) = (1 - \cos \alpha - \sin \alpha) / (2\sqrt{2})$$

where α parameterizes the extra degree of freedom.

Note:

$\alpha = 0, \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow$ Haar again

$\alpha = \pi \Rightarrow$ degenerate condition

other values of α give various length-4 scaling filters. In particular,

$\alpha = \frac{\pi}{3} \Rightarrow$ Daubechies' length-4 scaling filter

(more on this shortly)

Length - 6 Filters

linear + quadratic constraints

\rightarrow 4 equations in 6 unknowns

\Rightarrow 2 extra degrees of freedom

The solutions take the general form

$$h(0) = \left[(1 + \cos \alpha + \sin \alpha)(1 - \cos \beta - \sin \beta) + 2 \sin \beta \cos \alpha \right] / (4\sqrt{2})$$

$$h(1) = \left[(1 - \cos \alpha + \sin \alpha)(1 + \cos \beta - \sin \beta) - 2 \sin \beta \cos \alpha \right] / (4\sqrt{2})$$

$$h(2) = [1 + \cos(\alpha - \beta) + \sin(\alpha - \beta)] / (2\sqrt{2})$$

$$h(3) = [1 + \cos(\alpha - \beta) - \sin(\alpha - \beta)] / (2\sqrt{2})$$

$$h(4) = \frac{1}{\sqrt{2}} - h(0) - h(2)$$

$$h(5) = \frac{1}{\sqrt{2}} - h(1) - h(3)$$

where α, β are free parameters.

$$\alpha = \beta \Rightarrow \text{Haar}$$

$$\beta = 0, \alpha = \frac{\pi}{3} \Rightarrow \text{Daubechies' length-4}$$

$$\alpha = 1.3598\dots, \beta = -0.782106\dots$$

$$\Rightarrow \text{Daubechies' length-6}$$

It is difficult to obtain general forms for longer filters.

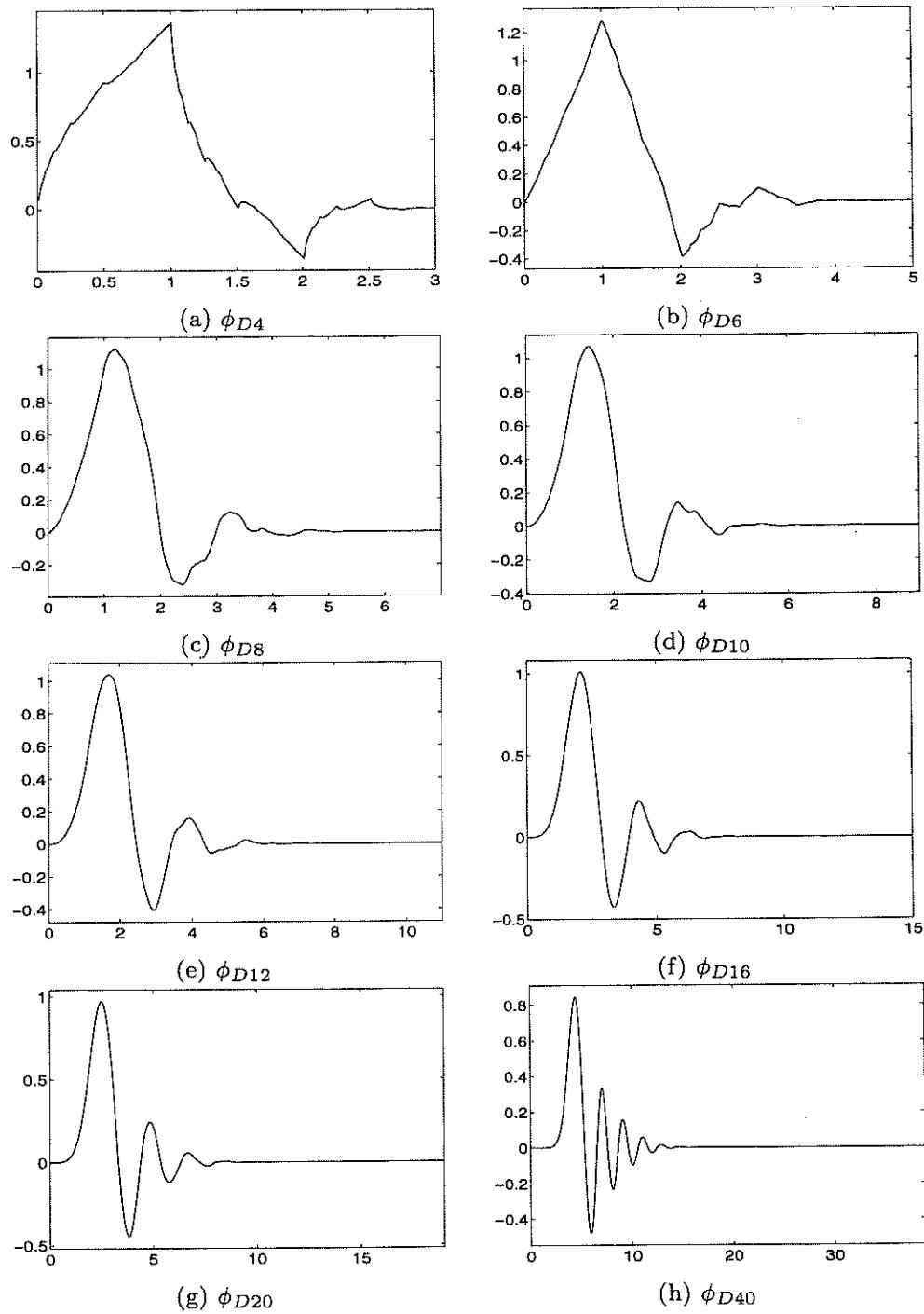


Figure 6.1. Daubechies Scaling Functions, $N = 4, 6, 8, \dots, 40$

freedom to maximize the differentiability of $\varphi(t)$ rather than maximize the zero moments. This is not easily parameterized, and it gives only slightly greater smoothness than the Daubechies system [Dau92].

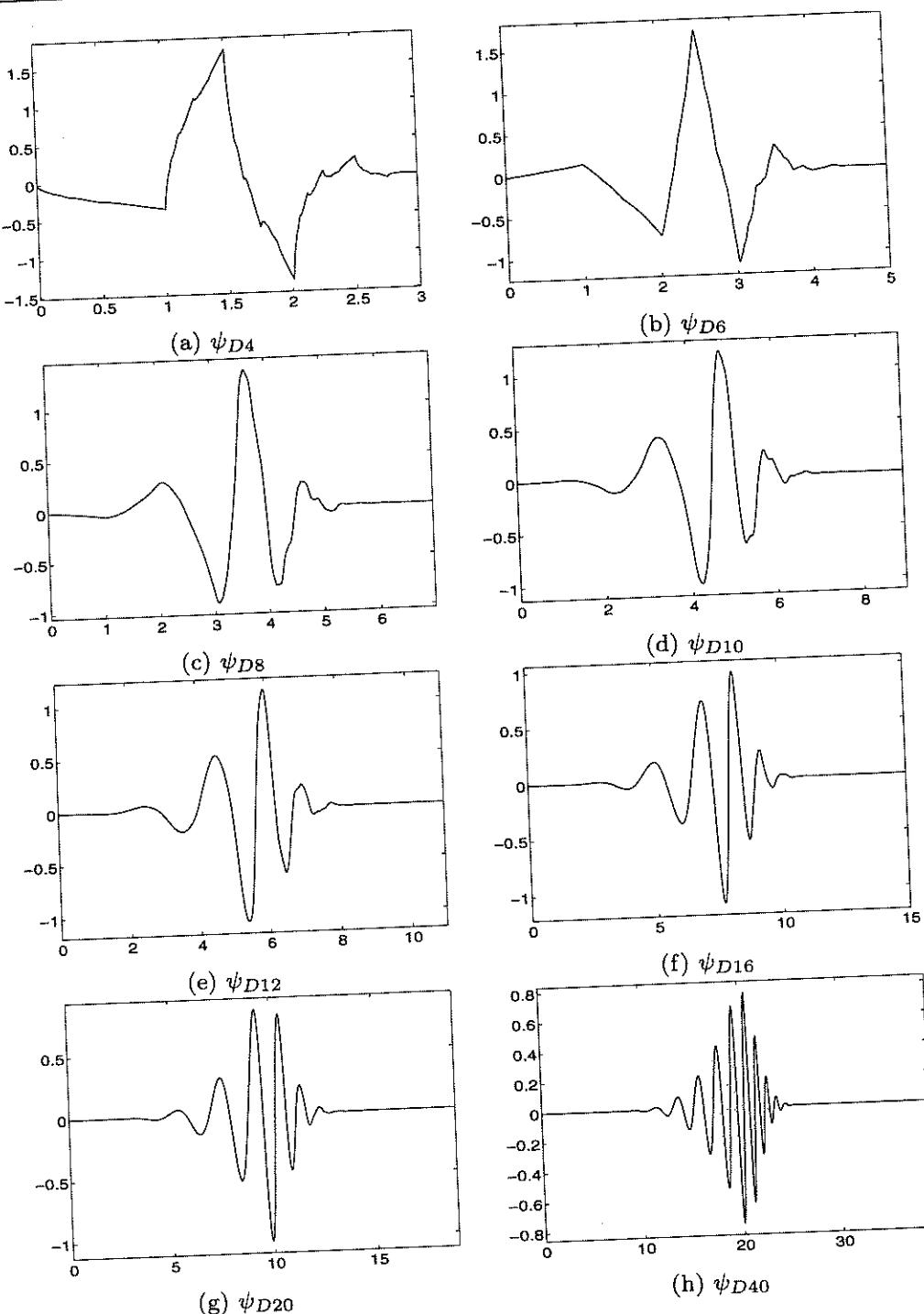


Figure 6.2. Daubechies Wavelets, $N = 4, 6, 8, \dots, 40$

Examples of Daubechies scaling functions resulting from choosing different factors in the spectral factorization of $|H(\omega)|^2$ in (6.18) can be found in [Dau92].

Wavelet Design and Vanishing Moments

Assume that $h(n)$ is a length- N scaling filter.

We then have one linear equation

$$\sum_{n=0}^{N-1} h(n) = \sqrt{2}$$

and $\frac{N}{2}$ quadratic equations

$$\sum_{n=0}^{N-1} h(n) h(n-2k) = \delta(k)$$

leaving us with $\frac{N}{2} - 1$ degrees of freedom in choosing $h(n)$.

We can use these extra degrees of freedom to obtain some desirable characteristics.

In particular, it turns out to be especially useful to design $h_i(n)$ so that the corresponding wavelet filter

$$h_i(n) = (-1)^{l-n} h(l-n)$$

has "vanishing moments".

The moments of $h_i(n)$ are

$$\mu_i(k) = \sum_n n^k h_i(n), \quad k=0, 1, \dots$$

If $\mu_i(k) = 0$, $k=0, 1, \dots, K-1$,

then we say that $h_i(n)$ has

K vanishing moments.

Why are vanishing moments important?

Define the following moments:

$$m(k) = \int t^k \phi(t) dt$$

$$m_1(k) = \int t^k \psi(t) dt$$

$$k=0, 1, 2, \dots$$

$$\mu(n) = \sum_n n^k h(n)$$

$$\mu_1(k) = \sum_n n^k h_1(k)$$

The continuous moments $m(k)$ and $m_1(k)$ can be related to the moments of the discrete filters, $\mu(n)$ and $\mu_1(k)$ using the scaling equation.

$$\phi(t) = \sum_n h(n) \sqrt{2} \phi(2t-n)$$

\Rightarrow

$$m(k) = \int t^k \phi(t) dt = \sum_n h(n) \sqrt{2} \int t^k \phi(2t-n) dt$$

$$= \sum_n h(n) \frac{\sqrt{2}}{2^{kn}} \int (u+n)^k \phi(u) du , \quad u=2t-n$$

$$= \sum_n h(n) \frac{\sqrt{2}}{2^{k+1}} \int \sum_{\ell=0}^k \binom{k}{\ell} u^{k-\ell} n^\ell \phi(u) du$$

$$\text{where } \binom{k}{\ell} = \frac{k!}{(k-\ell)! \ell!}$$

$$= \frac{\sqrt{2}}{2^{k+1}} \sum_{\ell=0}^k \binom{k}{\ell} \sum_n h(n) n^\ell \cdot \int u^{k-\ell} \phi(u) du$$

$$\Rightarrow m(k) = \frac{\sqrt{2}}{2^{k+1}} \sum_{\ell=0}^k \binom{k}{\ell} u(\ell) m(k-\ell)$$

Similarly

$$M_1(k) = \frac{\sqrt{2}}{2^{k+1}} \sum_{\ell=0}^k \binom{k}{\ell} u_1(\ell) m(k-\ell)$$

Vanishing Moments and Polynomial Approximation

Note:

$$m_1(k) = 0, \quad k=0, 1, \dots, K-1$$

$$\implies m_i(k) = 0, \quad k=0, 1, \dots, K-1$$

Now suppose the signal we are studying is a polynomial of degree $L-1$:

$$f(t) = \sum_{\ell=0}^{L-1} a_\ell t^\ell$$

If $L \leq K$, then

$$\begin{aligned}
 \langle f, \psi_{j,k} \rangle &= \int f(t) \psi(2^j t - k) z^{-j/2} dt \\
 &= z^{-j/2} \int f\left(\frac{u+k}{z^j}\right) \psi(u) du, \quad u = 2^j t - k \\
 &= z^{-j/2} \int \sum_{\ell=0}^{L-1} a_\ell \left(\frac{u+k}{z^j}\right)^\ell \psi(u) du \\
 &= \sum_{\ell=0}^{L-1} \tilde{a}_\ell \cdot \int u^\ell \psi(u) du = 0
 \end{aligned}$$

\Rightarrow All wavelet coefficients are zero!!

Wavelets with vanishing moments
are "blind" to polynomial
structure.

Polynomial functions can be
represented in terms of scaling
functions alone !!

Bottom Line:

If we have signals that are
well approximated by polynomials
(or piecewise polynomial functions),
then the wavelet transform will
be very, very sparse; most
coefficients will be zero.

Key to wavelet-based compression
and denoising

Frequency Domain Characterization

Let $\psi(t)$ be the basic wavelet

associated with $h(n)$ and $h_1(n) = (-1)^{l-n} h(l-n)$.

Also, let $\Psi(\omega)$, $H(\omega)$ and $H_1(\omega)$ denote
the Fourier transforms of ψ , h , and h_1 .

Theorem:

The following conditions are equivalent.

- (i) The wavelet ψ has K vanishing moments
- (ii) $H(\omega)$ has $K-1$ zeros at $\omega = \pi$

Proof:

Let $\Psi^{(k)}(\omega)$ denote the k -th derivative
of $\Psi(\omega)$. Recall

$$(-jt)^k \psi(t) \xleftrightarrow{\text{FT}} \Psi^{(k)}(\omega)$$

Thus,

$$\Psi^{(k)}(0) = \int (-j\omega)^k \psi(t) dt$$
$$= 0 \quad \text{for } k \leq K-1$$

Now, from the scaling equation
we have

$$\frac{1}{r_2} \psi(\frac{t}{r_2}) = \sum_n (-1)^{1-n} h(1-n) \phi(t-n).$$

Taking the Fourier transform of
both sides:

$$\begin{aligned} r_2 \Psi(z\omega) &= \sum_n (-1)^{1-n} h(1-n) \int \phi(t-n) e^{-j\omega t} dt \\ &= \sum_n (-1)^{1-n} h(1-n) \int \phi(u) e^{-j\omega(u+n)} du \\ &= \overline{\Phi}(\omega) \cdot \sum_n (-1)^{1-n} h(1-n) e^{-j\omega n} \\ &\quad \xrightarrow{\substack{\text{FT of } \Phi(t)}} \\ &= \overline{\Phi}(\omega) \cdot \sum_m (-1)^m h(m) e^{-j\omega(1-m)} \\ &= \overline{\Phi}(\omega) \cdot \sum_m (-1)^m h(m) e^{-j\omega} \cdot e^{j\omega m} \end{aligned}$$



$$\begin{aligned}\boxed{\mathcal{F}_2 \Psi(2\omega) &= \Phi(\omega) \cdot e^{-j\omega} \cdot \sum_m h(m) e^{j(\omega+\pi)m} \\ &= \Phi(\omega) \cdot e^{-j\omega} H^*(\omega+\pi)}\end{aligned}$$

Now if ψ has K vanishing moments
and thus $\Psi^{(k)}(0) = 0, k=0, 1, \dots K-1,$

then

$$\left. \frac{\partial^k H(\omega)}{\partial \omega^k} \right|_{\omega=\pi} = 0, \quad k=0, 1, \dots K-1 \quad \textcircled{4}$$

Conversely, if $\textcircled{4}$ holds, then

$$\Psi^{(k)}(0) = 0, \quad k=0, 1, \dots K-1.$$



This relation is central to the
Daubechies' method of wavelet
design.