# Perfect Reconstruction Filter Banks and Multiresolution Analysis

In this lecture we will:

- determine conditions that guarantee the filter bank implementation of the DWT is invertible

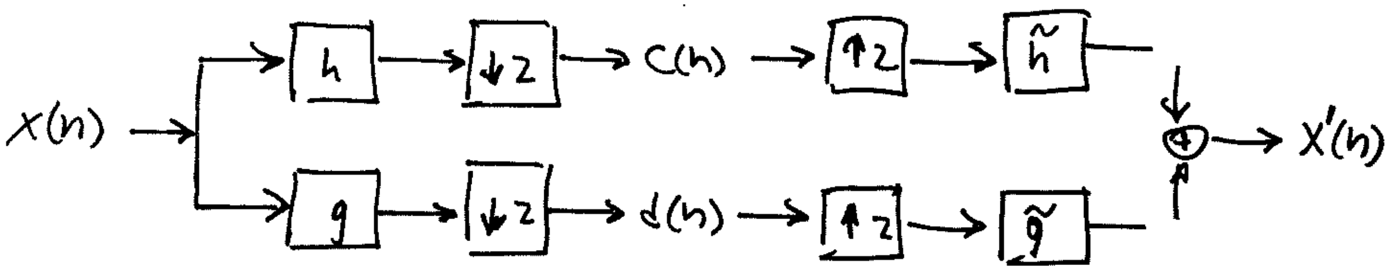
- relate the wavelet filter to the scaling filter

- establish necessary & sufficient conditions for the scaling filter that guarantee a valid MRA.

- relate scaling and wavelet filters to the corresponding continuous scaling and wavelet functions

# Perfect Reconstruction Filter Banks

Consider a simple, two-channel filter bank:




We want $x'(n) = x(n)$ <u>Perfect Reconstruction</u>

What conditions must the filters $(h, g, \tilde{h}, \tilde{g})$ satisfy to guarantee PR?

<u>Note</u>: $\tilde{h} = h$ <u>scaling filter</u>

$g = \tilde{g} = h_1$ <u>wavelet filter</u>

for DWT

<u>Note</u>: $x(n) \longrightarrow \boxed{h} \longrightarrow y(n)$

$\equiv \quad y(n) = x * h(-n)$ (time-reversed filter)

## Theorem I (Vetterli)

The filterbank performs an exact reconstruction for every input signal if and only if

$$H^*(\omega+\pi)\,\tilde{H}(\omega) + G^*(\omega+\pi)\,\tilde{G}(\omega) = 0$$

and

$$H^*(\omega)\,\tilde{H}(\omega) + G^*(\omega)\,\tilde{G}(\omega) = 2$$

**proof:**

First consider

$$x(n) \longrightarrow \boxed{h} \xrightarrow{c'(n)} \boxed{\downarrow 2} \longrightarrow c(n)$$

The Fourier transform of $c(n)$ can be written as

$$C(2\omega) = \sum_n c'(2n)\, e^{-j2\omega n}$$

or

$$C(2\omega) = \sum_n c'(2n)\, e^{-j2(\omega+\pi)n} \quad , \text{ since } e^{-j2\pi n} = 1$$

So, we have

$$C(2\omega) = \frac{1}{2}\left(\sum_n c'(2n)e^{-j2\omega n} + \sum_n c'(2n)e^{-j2(\omega+\pi)n}\right)$$

$$= \frac{1}{2}\left(C'(\omega) + C'(\omega+\pi)\right)$$

$$= \frac{1}{2}\left(X(\omega)H^*(\omega) + X(\omega+\pi)H^*(\omega+\pi)\right)$$

conjugation due to time reversal of filter

Similarly,

$$D(2\omega) = \frac{1}{2}\left(X(\omega)G^*(\omega) + X(\omega+\pi)G^*(\omega+\pi)\right)$$

Next, we up-sample $c(n)$ and $d(n)$

$$c''(n) = \begin{cases} c(p), & n = 2p \\ 0, & \text{otherwise} \end{cases}$$

$$d''(n) = \begin{cases} d(p), & n = 2p \\ 0, & \text{o.w.} \end{cases}$$

Note that

$$C''(\omega) = \sum_n c''(n) e^{-j\omega n}$$

$$= \sum_n c(n) e^{-j\omega 2n}$$

$$= C(2\omega)$$

Similarly, $D''(\omega) = D(2\omega)$.

Finally, the up-sampled signals $c''(n)$ and $d''(n)$ are filtered by $\tilde{h}$ and $\tilde{g}$, resp. Thus, the output $x'(n)$ has DTFT

$$X'(\omega) = C(2\omega)\tilde{H}(\omega) + D(2\omega)\tilde{G}(\omega)$$

$$= \frac{1}{2}\left[X(\omega)H^*(\omega) + X(\omega+\pi)H^*(\omega+\pi)\right]\tilde{H}(\omega)$$

$$+ \frac{1}{2}\left[X(\omega)G^*(\omega) + X(\omega+\pi)G^*(\omega+\pi)\right]\tilde{G}(\omega)$$

Re-arranging terms,

$$X'(\omega) = \frac{1}{2}\left[ H^*(\omega)\,\tilde{H}(\omega) + G^*(\omega)\,\tilde{G}(\omega) \right] X(\omega)$$
$$+ \frac{1}{2}\left[ H^*(\omega+\pi)\,\tilde{H}(\omega) + G^*(\omega+\pi)\,\tilde{G}(\omega) \right] X(\omega+\pi)$$

We want $X'(\omega) = X(\omega)$, so we require

$$H^*(\omega)\,\tilde{H}(\omega) + G^*(\omega)\,\tilde{G}(\omega) = 2$$

and

$$H^*(\omega+\pi)\,\tilde{H}(\omega) + G^*(\omega+\pi)\,\tilde{G}(\omega) = 0$$

$\blacksquare$

In fact, the previous theorem shows that the synthesis filters $\tilde{h}$ and $\tilde{g}$ are determined by the analysis filters $h$ and $g$, as shown by the following argument.

In matrix form, the PR conditions become,

$$\begin{bmatrix} H(\omega) & G(\omega) \\ H(\omega+\pi) & G(\omega+\pi) \end{bmatrix} \begin{bmatrix} \tilde{H}^*(\omega) \\ \tilde{G}^*(\omega) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Inverting this equation gives

$$\begin{bmatrix} \tilde{H}^*(\omega) \\ \tilde{G}^*(\omega) \end{bmatrix} = \frac{2}{\Delta(\omega)} \begin{bmatrix} G(\omega+\pi) \\ -H(\omega+\pi) \end{bmatrix} \quad \star$$

where $\Delta(\omega)$ is the determinant

$$\Delta(\omega) = H(\omega)\,G(\omega+\pi) - H(\omega+\pi)\,G(\omega).$$

When all filters are FIR,

$\Delta(\omega)$ can be evaluated directly, and simpler relations between the analysis and synthesis filters can be obtained.

<u>Theorem 2</u> (p. 262 Mallat)

Perfect reconstruction filters satisfy

$$H^*(\omega)\,\tilde{H}(\omega) + H^*(\omega+\pi)\,\tilde{H}(\omega+\pi) = 2. \quad (\#)$$

For FIR filters, there exists $\alpha \in \mathbb{R}$ and $l \in \mathbb{Z}$ such that

$$G(\omega) = \alpha e^{-i(2l+1)\omega}\,\tilde{H}^*(\omega+\pi)$$

$$\tilde{G}(\omega) = \alpha^{-1} e^{-i(2l+1)\omega}\,H^*(\omega+\pi).$$

<u>proof:</u>

The matrix inverse equation $\ast$ shows that

$$\tilde{H}^*(\omega) = \frac{2}{\Delta(\omega)}\, G(\omega+\pi)$$

$$\tilde{G}^*(\omega) = \frac{-2}{\Delta(\omega)}\, H(\omega+\pi)$$

Therefore,

$$G(\omega)\,\tilde{G}^*(\omega) = -\frac{\Delta(\omega+\pi)}{\Delta(\omega)}\, \tilde{H}^*(\omega+\pi)\, H(\omega+\pi)$$

Note that

$$\Delta(\omega+\pi) = H(\omega\ \pi)\, G(\omega) - H(\omega\ )\, G(\omega+\pi)$$

since $H, G$ are $2\pi$-periodic

$$= -\Delta(\omega)$$

$\Rightarrow$

$$G(\omega)\,\tilde{G}^*(\omega) = \tilde{H}^*(\omega+\pi)\, H(\omega+\pi)$$

From Thm 1, $H^*(\omega)\,\tilde{H}(\omega) + G^*(\omega)\,\tilde{G}(\omega) = 2$

$$\Rightarrow H^*(\omega)\,\tilde{H}(\omega) + \tilde{H}^*(\omega+\pi)\, H(\omega+\pi) = 2 .$$

Now, to obtain a relation between $g, \tilde{g}$ and $h, \tilde{h}$ we note the following.

(i) the DTFT of an FIR filter is a finite series in $e^{\pm in\omega}$

(ii) if all filters are FIR, then $\Delta(\omega)$ is a finite series

(iii) since $\tilde{H}(\omega)$ and $\tilde{G}(\omega)$ must be finite series, the facts

$$\tilde{H}^*(\omega) = \frac{2}{\Delta(\omega)} G(\omega+\pi), \quad \tilde{G}^*(\omega) = \frac{-2}{\Delta(\omega)} H(\omega+\pi)$$

imply that

$\frac{1}{\Delta(\omega)}$ is also a finite series.

However, the only DTFT whose inverse and itself are both finite series is a one-term series:

$$\Rightarrow \quad \Delta(\omega) = \beta e^{i\omega m}$$

$$\frac{1}{\Delta(\omega)} = \frac{1}{\beta} e^{-i\omega m}$$

Furthermore, since

$$\Delta(\omega) = -\Delta(\omega+\pi)$$

the exponent $m$ must be odd.

Thus,
$$\Delta(\omega) = \beta\, e^{i(2\ell+1)\omega}$$

$$\Rightarrow \quad G(\omega) = \frac{\beta e^{i(2\ell+1)\omega}}{2}\, \tilde{H}^*(\omega+\pi)$$

$$\tilde{G}(\omega) = \frac{-2}{\beta e^{-i(2\ell+1)\omega}}\, H^*(\omega+\pi)$$

$$= -\left(\frac{2}{\beta}\right) e^{i(2\ell+1)\omega}\, H^*(\omega+\pi)$$

Take $\alpha = \frac{\beta}{2}$ to finish the proof.

$\blacksquare$

The values $\alpha \in \mathbb{R}$ and $\ell \in \mathbb{Z}$ are arbitrary. In particular, with $\alpha = 1$ and $\ell = 0$ we obtain

$$G(\omega) = e^{-i\omega} \tilde{H}^*(\omega + \pi)$$

$$\overset{DTFT}{\longleftrightarrow} \quad g(n) = (-1)^{1-n} \tilde{h}(1-n)$$

$$\tilde{G}(\omega) = e^{-i\omega} H^*(\omega + \pi)$$

$$\overset{DTFT}{\longleftrightarrow} \quad \tilde{g}(n) = (-1)^{1-n} h(1-n)$$

<u>Special Case:</u> DWT $h = \tilde{h}$

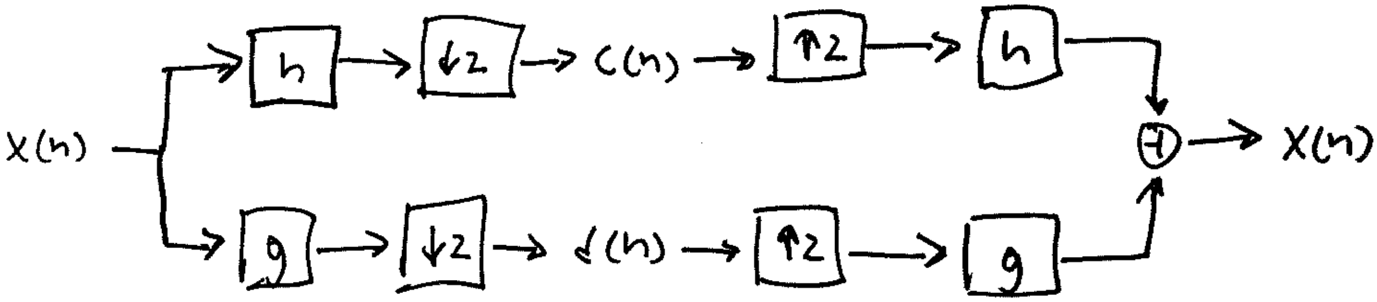
Smith-Barnwell Condition

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$$

(quadrature mirror filter)

$$g(n) = (-1)^{1-n} h(1-n)$$

# Filter Banks and DWT




$$g(n) = (-1)^{(1-n)}\, h(1-n) \qquad \underline{\text{FIR}}$$

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$$

(PR condition)

Does the iterated filter bank produce a valid MRA?

Theorem 3 (Time-domain PR condition)

For any FIR $h(n)$

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$$

$$\Leftrightarrow \quad \sum_n h(n)h(n-2k) = \delta(k)$$

proof:

Let $y(k) = \sum_n h(n)h(n-2k)$

$$Y(2\omega) = \sum_k \left(\sum_n h(n)h(n-2k)\right) e^{-j2\omega k}$$

$$= \sum_n h(n) \sum_k h(n-2k) e^{-j\omega 2k}$$

$$= \sum_n h(n) \sum_m h(m) e^{-j\omega(n-m)}$$

$$= H(\omega) \cdot H^*(\omega)$$

$$= |H(\omega)|^2$$

Also, since DTFTs are $2\pi$-periodic

$$Y(2\omega) = Y(2(\omega+\pi))$$

$$= |H(\omega+\pi)|^2$$

So, we can write

$$Y(2\omega) = \frac{1}{2}\left( |H(\omega)|^2 + |H(\omega+\pi)|^2 \right)$$

$$= 1 \quad \text{(according to PR condition)}$$

$$\Rightarrow \quad y(n) = \delta(n)$$

Alternatively,

if $\sum h(n) h(n-2k) = \delta(k)$,

then $|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2.$

$\blacksquare$

# Perfect Reconstruction and MRA

It turns out that the PR condition

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$$

or equivalently

$$\sum_n h(n)\, h(n-2k) = \delta(k)$$

is also key to establishing that $h(n)$ generates a valid MRA and an orthonormal wavelet basis for $L^2(\mathbb{R})$.

# Theorem 4 (Cohen)

Let $h(n)$ be an FIR scaling filter and let $H(\omega)$ be its DTFT.

If

(i) $\sum_n h(n) = \sqrt{2}$

(ii) $\sum_n h(n)\, h(n-2k) = \delta(k)$

(iii) $H(\omega) \neq 0$ for $-\frac{\pi}{3} \leq \omega \leq \frac{\pi}{3}$

Then the scaling equation

$$\phi(t) = \sum_n h(n) \sqrt{2}\, \phi(2t-n)$$

has a unique solution $\phi \in L^2(\mathbb{R})$ and $\{\phi(t-k)\}$ are orthogonal.

Moreover, (i), (ii), and (iii) guarantee that $h(n)$ produces an orthogonal MRA of $L^2(\mathbb{R})$.

proof:

To show that (i), (ii), and (iii) are <u>sufficient</u> to guarantee an orthogonal MRA takes a lot of work (see A. Cohen's book).

So, here we will just establish that (i) and (ii) are <u>necessary</u> conditions. This illustrates the importance of these two key properties of the scaling filter.

Necessity of (i):

Let $\phi$ be a solution to the scaling equation. Then

$$\phi(t) = \sum_n h(n) \sqrt{2}\, \phi(2t-n)$$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(t)\, dt = \int_{-\infty}^{\infty} \sum_n h(n) \sqrt{2}\, \phi(2t-n)\, dt$$

Interchanging summation and integration

$$\int_{-\infty}^{\infty} \phi(t)\,dt = \sum_n h(n) \sqrt{2} \int_{-\infty}^{\infty} \phi(2t-n)\,dt$$

$$= \sum_n h(n)\sqrt{2} \int_{-\infty}^{\infty} \phi(x)\, \tfrac{1}{2}\, dx \quad , \; x = 2t$$

$$\Rightarrow \qquad 1 = \frac{1}{\sqrt{2}} \sum_n h(n)$$

or $\boxed{\sum_n h(n) = \sqrt{2}}$

Thus, if $\phi$ is a solution of the scaling equation, $h(n)$ must satisfy (i).

Necessity of (ii):

Let $\phi$ be a solution of the scaling equation and furthermore assume that $\{\phi(t-n)\}$ are orthonormal.

Orthogonality means

$$\int \phi(t)\,\phi(t-k)\,dt = 0 \qquad \text{for } k \neq 0 \qquad (\star)$$

$$\Rightarrow \int \sum_n h(n)\sqrt{2}\,\phi(2t-n) \cdot \sum_m h(m)\sqrt{2}\,\phi(2t-2k-m)\,dt = 0$$

$$k \neq 0$$

$$\Rightarrow \sum_m \sum_n h(n)\,h(m) \int \phi(x-2k-m)\,\phi(x-n)\,dx = 0 \qquad (\#)$$

where $x = 2t$.

$(\star)$ also implies

$$\int \phi(x-2k-m)\,\phi(x-n)\,dx = \delta(n-2k-m)$$

So, $(\#)$ becomes

$$\boxed{\sum_n h(n)\,h(n-2k) = 0 \qquad k \neq 0.}$$

$\blacksquare$

Summary:

(i) $\sum_n h(n) = \sqrt{2}$

is necessary for any solution to the scaling equation

$$\phi(t) = \sum_n h(n)\sqrt{2}\,\phi(2t-n)$$

(ii) $\sum h(n)\,h(n-2k) = \delta(k)$

is necessary to ensure that $\{\phi(t-n)\}$ are orthonormal.

Also, this is equivalent to PR condition $|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$

(iii) $H(\omega) \neq 0$ on $-\frac{\pi}{3} \le \omega \le \frac{\pi}{3}$

is a extra requirement needed for proving that $h(n)$ will generate a valid MRA.

It basically says that $h(n)$ must be a lowpass filter.