# Filter Banks and the DWT

In many practical applications we begin with a set of signal samples $\{f(k2^{-J})\}_{k=0}^{2^J-1}$, which may be regarded (approximately) as scaling coefficients at scale $2^{-J}$.

In such situations, we never have to deal with scaling or wavelet functions directly — all analysis can be carried out with digital filters.

In fact, the DWT can be computed using a very simple filter bank structure.

# Analysis (Fine-to-Coarse)

The basic recursion relating scaling functions $\phi(t)$ and $\{\phi(2t-n)\}$

$$\phi(t) = \sum_n h(n)\sqrt{2}\,\phi(2t-n)$$

leads to the recursive expressions for scaling and wavelet coefficients

$$c_j(k) = \sum_m h(m-2k)\, c_{j+1}(m)$$

$$d_j(k) = \sum_m h_1(m-2k)\, c_{j+1}(m)$$

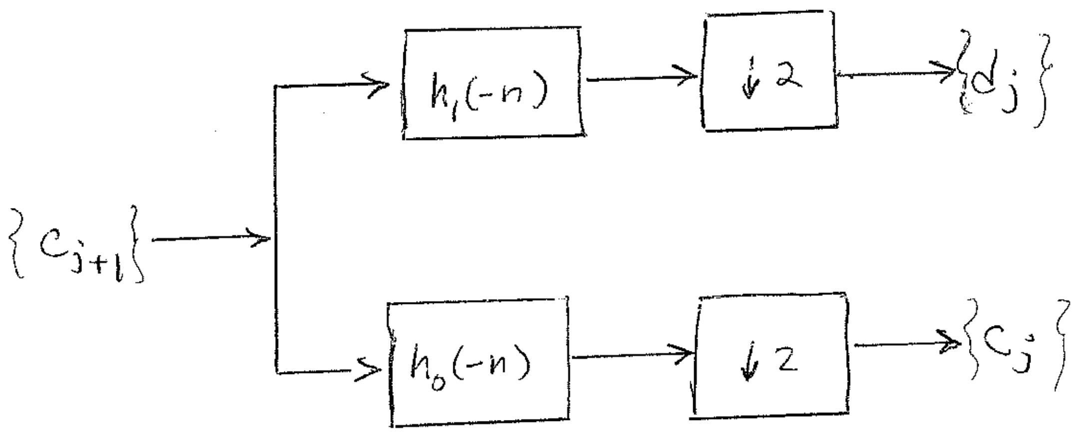
We can interpret these calculations as a filter bank (two-band) consisting of filters $\{h(n)\}$ and $\{h_1(n)\}$ and downsamplers.

# Downsampling (Decimation)

$$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow x(2n)$$

(only even indexed samples at output)

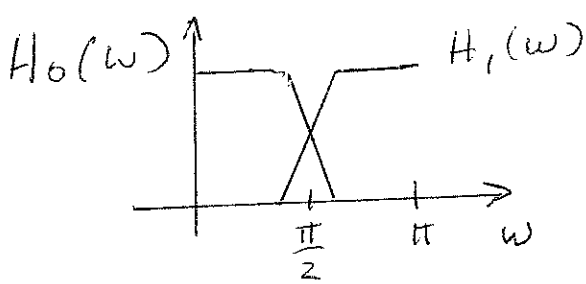
# Two-Band Analysis Bank




where $h_0(n) = h(n)$

Recall, $h_0 \approx$ lowpass filter

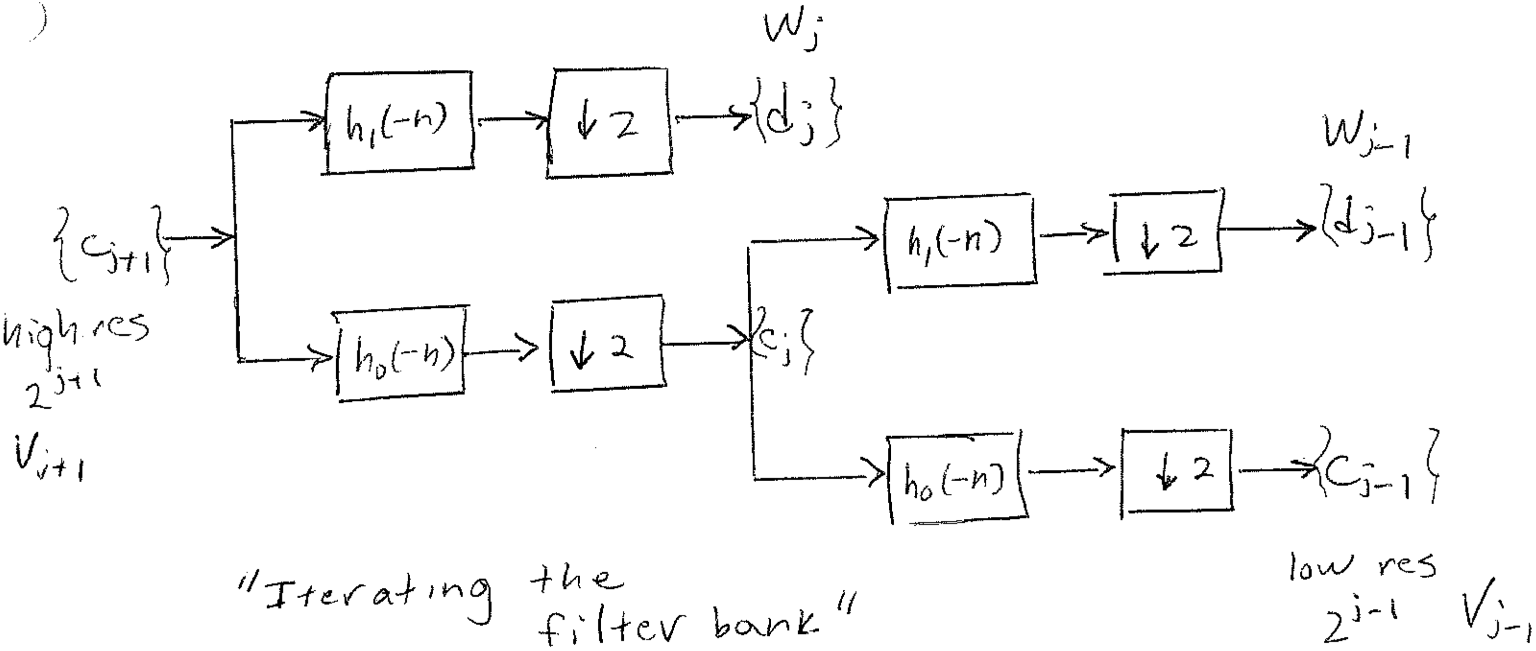
$h_1 \approx$ highpass filter




$$H_i(\omega) = \sum_{n=-\infty}^{\infty} h_i(n) e^{i\omega n}$$

<u>DTFT of $h_i(n)$</u>

# Two-Stage Two-Band Analysis Tree



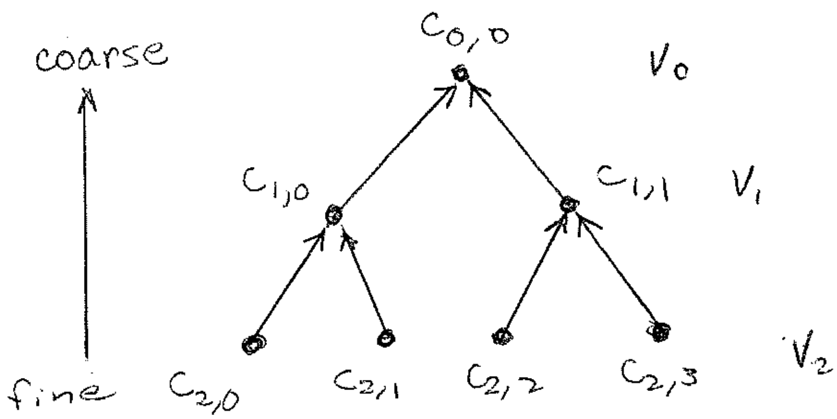

"Iterating the filter bank"

fine $\longrightarrow$ coarse

## Graphical Representation

$c_{j,k} \equiv c_j(k)$

Take $j=1$, $\underline{c_2} = (c_{2,0}, c_{2,1}\ c_{2,2}\ c_{2,3})$




Binary tree representing multiscale analysis

The "edges" in the graph link nodes (scaling coefficients) corresponding to related "pieces" of the signal. Arrows indicate flow of analysis process.

# Frequency Bands of Analysis Tree

$$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow x(2n) \equiv y(n)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{i\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{-i\omega n}$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n)\, e^{i\omega n}$$

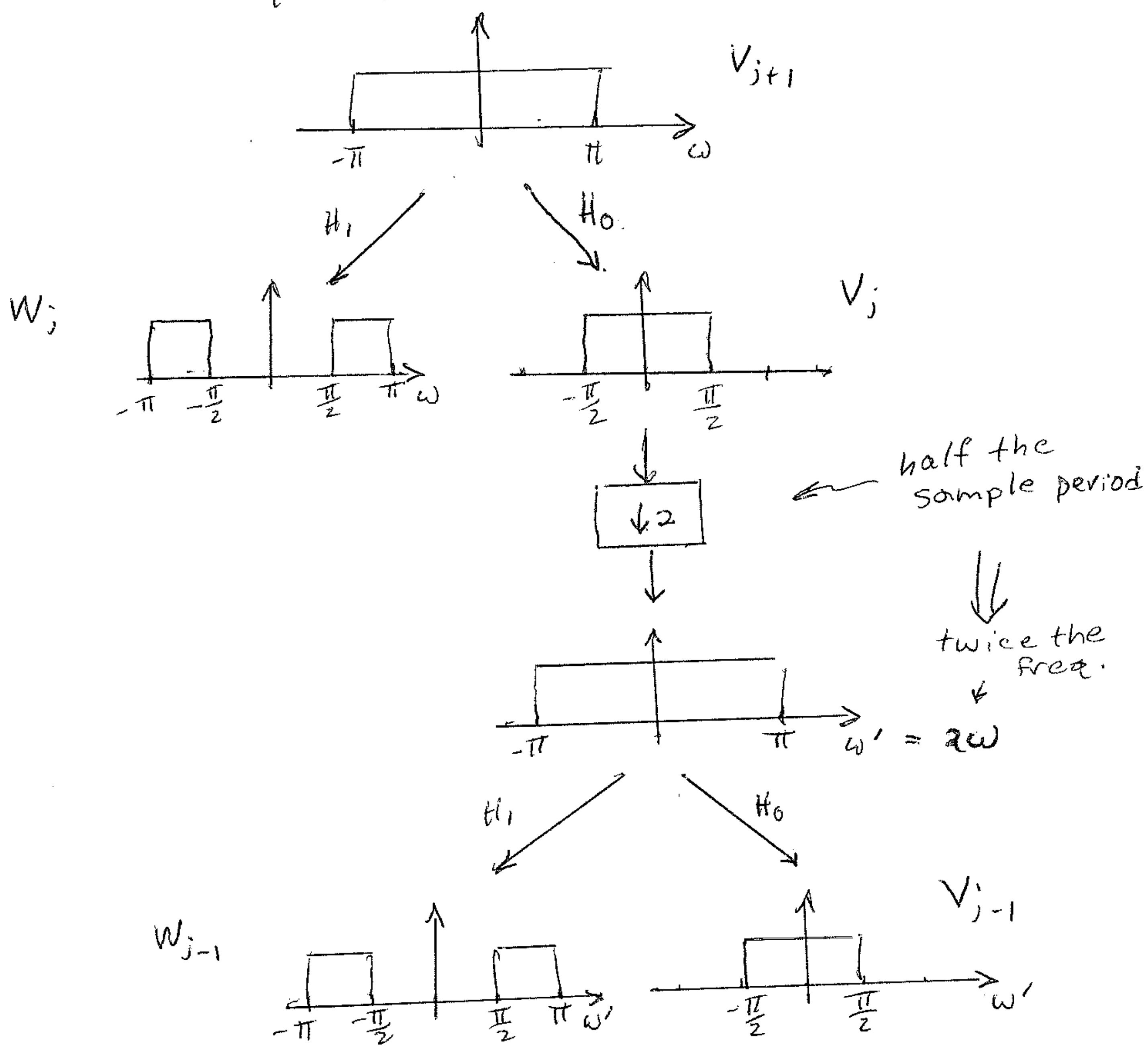
$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega')\, e^{-i\omega'(2n)}\, d\omega'\, e^{i\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega') \underbrace{\left( \sum_{n=-\infty}^{\infty} e^{-i2\omega' n}\, e^{i\omega n} \right)}_{\substack{\text{DTFT of } e^{i2\omega' n} \\ \delta(\omega - 2\omega')}} d\omega'$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega')\, \delta(\omega - 2\omega')\, d\omega' = X\left(\frac{\omega}{2}\right)$$

This implies the following frequency band decomposition.

Suppose $\{c_{j+1}(k)\}$ has support on $[-\pi, \pi]$ in frequency



So

$$V_{j+1} \approx \omega \in [-\pi, \pi]$$

$$W_j \approx \omega \in [-\pi, -\tfrac{\pi}{2}] \cup [\tfrac{\pi}{2}, \pi]$$

$$V_j \approx \omega \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

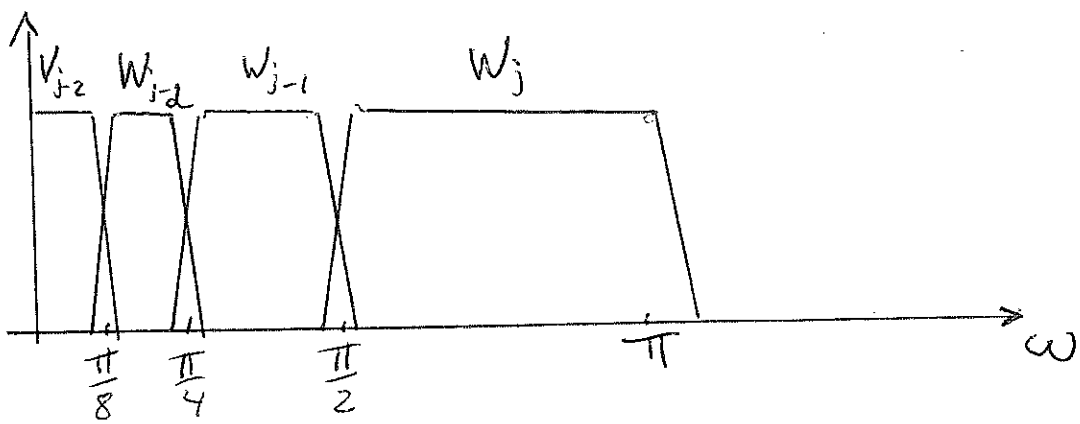
$$W_{j-1} \approx \omega' \in [-\pi, -\tfrac{\pi}{2}] \cup [\tfrac{\pi}{2}, \pi]$$

$$\equiv \omega \in [-\tfrac{\pi}{2}, -\tfrac{\pi}{4}] \cup [\tfrac{\pi}{4}, \tfrac{\pi}{2}]$$

$$V_{j-1} \approx \omega' \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\equiv \omega \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}]$$

$\vdots$



Wavelet subspaces are (approximately) bandpass subspaces.

Note that we have a logarithmic (base 2) set of bandwidths.

In filter bank parlance, we say that the filters ($h_0$ or $h_1$ in combo with $\downarrow 2$) are "constant-Q" filters because the bandwidths are proportional to the center frequency.

The logarithmic frequency decomposition effected by the DWT is similar to the octave decomposition in musical scales and is also related to the response characteristics of the human ear.

Synthesis (Coarse-to-fine)

Recall that if $f \in V_{j+1}$, then

1. $$f(t) = \sum_k c_{j+1}(k)\, 2^{(j+1)/2} \phi(2^{j+1}t - k)$$

$$= \sum_k c_j(k)\, 2^{j/2} \phi(2^j t - k)$$

$$+ \sum_k d_j(k)\, 2^{j/2} \psi(2^j t - k)$$

Also recall,

$$\phi(t) = \sum_n h(n) \sqrt{2}\, \phi(2t - n)$$

$$\Longrightarrow \phi(2^j t - k) = \sum_n h(n)\, 2^{(j+1)/2} \phi(2^{j+1}t - 2k - n)$$

$$\psi(t) = \sum_n h_1(n) \sqrt{2}\, \phi(2t - n)$$

$$\Longrightarrow \psi(2^j t - k) = \sum_n h_1(n)\, 2^{(j+1)/2} \phi(2^{j+1}t - 2k - n)$$

This shows that $f \in V_{j+1}$ can also be written as

(2) $$f(t) = \sum_k c_j(k) \sum_n h(n)\, 2^{(j+1)/2} \phi(2^{j+1}t - 2k - n)$$
$$+ \sum_k d_j(k) \sum_n h_1(n)\, 2^{(j+1)/2} \phi(2^{j+1}t - 2k - n)$$

Since $\{ 2^{(j+1)/2} \phi(2^{j+1}t - n)\}$ are o.n.

(1) $\Rightarrow \left\langle f, 2^{(j+1)/2}\phi(2^{j+1}t - k)\right\rangle = c_{j+1}(k)$

(2) $\Rightarrow \left\langle f, 2^{(j+1)/2}\phi(2^{j+1} - k)\right\rangle$

$$= \sum_m c_j(m)\, h(k-2m)$$
$$+ \sum_m d_j(m)\, h_1(k-2m)$$

$\Rightarrow$

$$\boxed{c_{j+1}(k) = \sum_m c_j(m)\, h(k-2m) + \sum_m d_j(m)\, h_1(k-2m)}$$

The filtering operations involved in synthesis can be viewed as a filter bank as well, involving the filters $h_0$ and $h_1$ and another processor called an <u>upsampler</u> (interpolation).

<u>Ex.</u>

$$\sum_m c_j(m)\, h(k-2m) = \sum_n \tilde{c}_j(n)\, h(k-n)$$

where

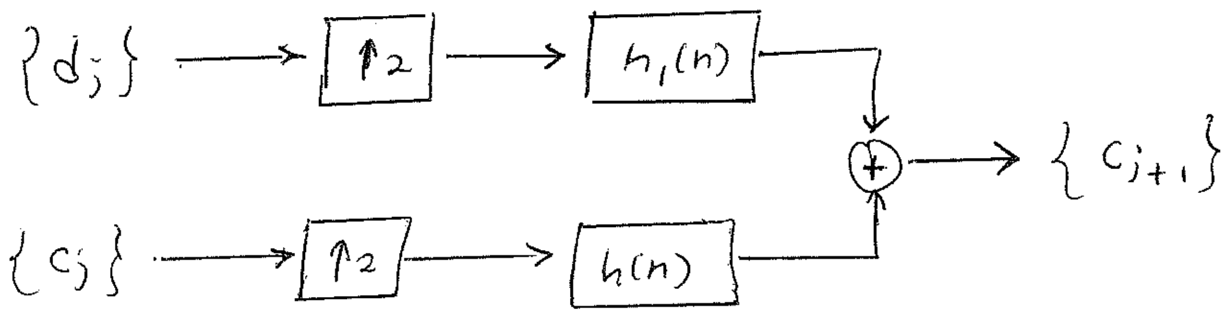
$$\tilde{c}_j(2n) = c_j(m)$$
$$\tilde{c}_j(2n+1) = 0$$

<u>Upsampler</u>

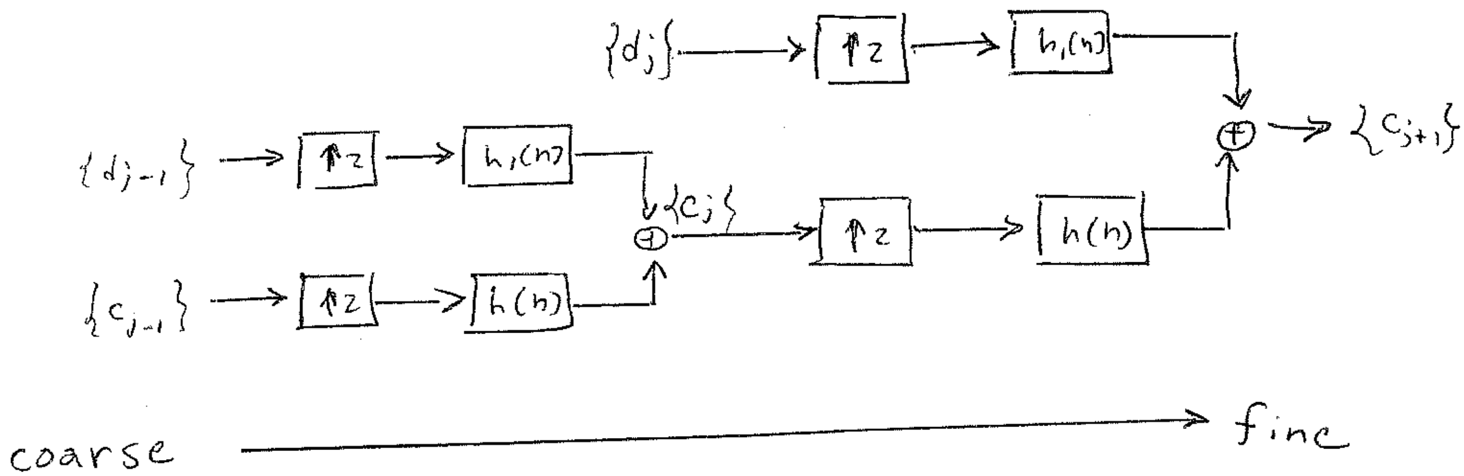
$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow y(n)$$

$$y(2n) = x(n)$$
$$y(2n+1) = 0$$

# Two-Band Synthesis Bank




# Two-Stage Two-Band Synthesis Tree



# Idealized Tiling of TF-plane

The energy of the wavelet basis function

$$2^{j/2}\,\psi(2^j t - k)$$

is concentrated about $(2^{-j}k,\ 2^j) = (t, f)$

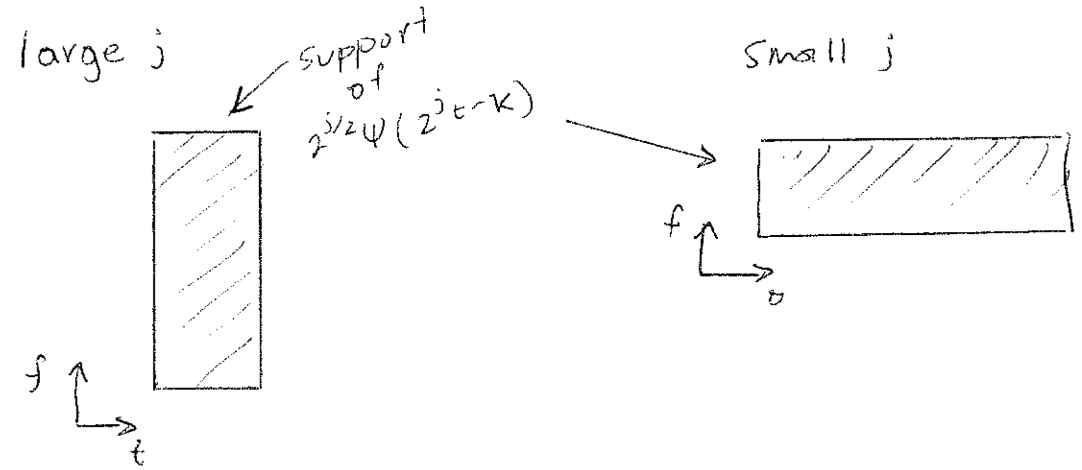
larger $j \Rightarrow$ higher resolution, smaller support in time

smaller $j \Rightarrow$ lower resolution, larger support in time.

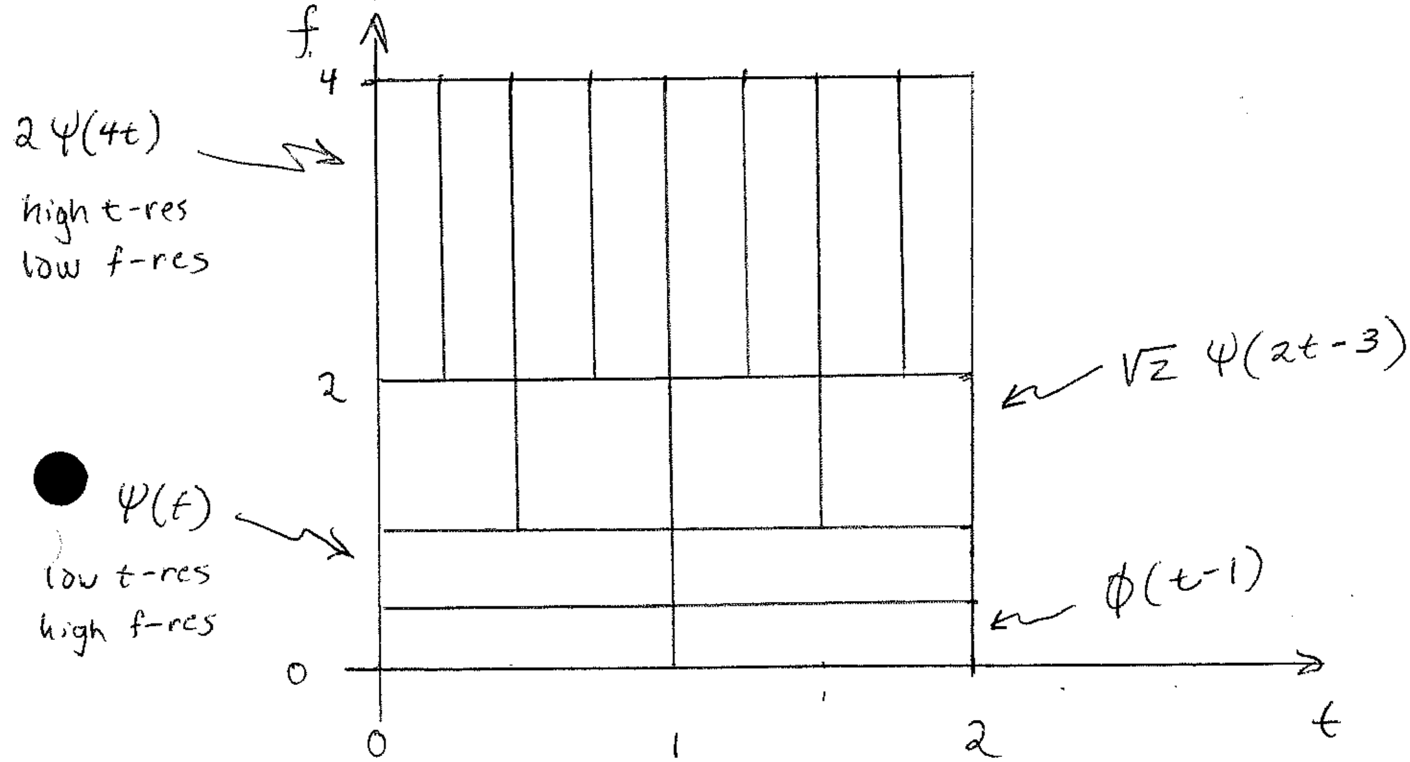
Also, the frequency decomposition effected by the DWT

$\Rightarrow$ larger $j \Rightarrow$ larger freq support

smaller $j \Rightarrow$ smaller freq support



This means that the joint time-freq analysis performed by the DWT can be visualized as follows,




each "tile" depicts the (idealized) support of the corresponding basis function. In other words, the wavelet or scaling coefficient associated with each basis function reflects the info in a signal in that time and freq range.

# Other Issues

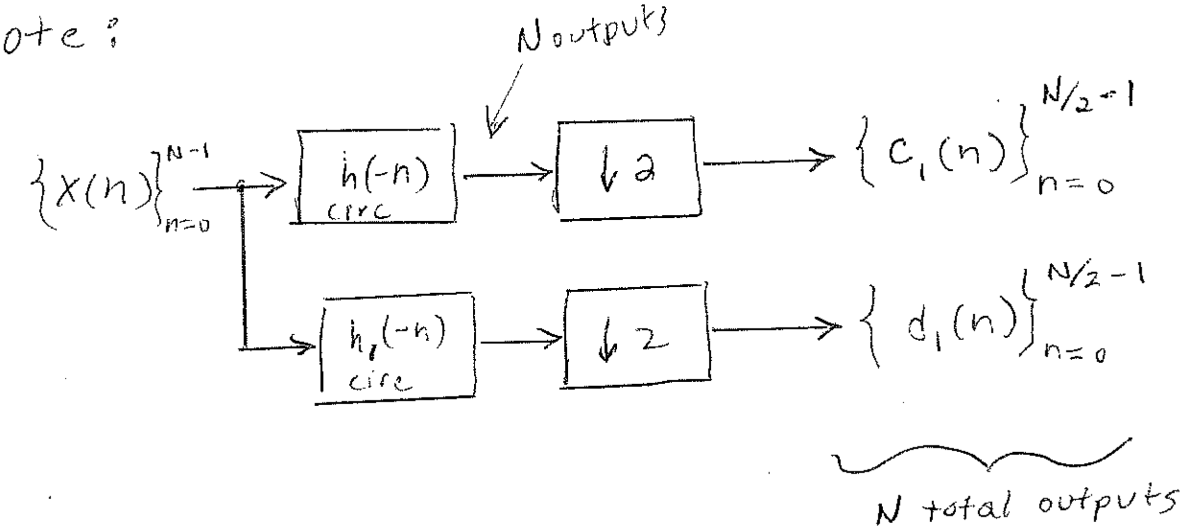
## Periodic vs. Nonperiodic DWTs

So far, we have assumed standard convolutions in the DWT filter bank. This implies that the output sequences are longer than the input sequences in general. (This issue does not arise in the case of the Haar because the filters are length 2 and the outputs are decimated).

For finite length signals (and images), a periodic version of the DWT is often used because it produces the same number of outputs (coefficients) as inputs (scaling coefficients or signal samples)

The periodic DWT is obtained by replacing regular convolutions in the filter bank with periodic (circular) convolutions. This is analogous to the (implicit) periodic extension of the signal encountered when processing with the DFT (FFT).

Note:




Periodic DWT can cause aliasing and "boundary" artifacts, but practically speaking there is little difference between it and the regular DWT.

Just as in the FFT, boundary artifacts can be overcome with zero padding (adding extra zeros to signal sequence so that circ conv equals regular conv).

## Numerical Complexity

· The circular convolution of an N-point sequence with a length $L$ filter requires

$$LN \text{ operations}$$

and since $L$ is usually small relative to $N$ we say it is $O(N)$ operations.

A decimated circular convolution of an N-pt sequence only requires

$$O\left(\frac{N}{2}\right) \text{ ops.}$$

level $j = J$ $\{x(n) = c_0(n)\}_{n=0}^{2^J - 1}$ $\underline{N = 2^J}$

level $j = J-1$ $\{d_1(n)\}_{n=0}^{2^{J-1}}$ : $O\left(\frac{N}{2}\right)$ ops

level $j = J-2$ $\{d_2(n)\}_{n=0}^{2^{J-2}}$ : $O\left(\frac{N}{4}\right)$ ops

$\vdots$

level $j$ $\{d_j(n)\}_{n=0}^{2^j}$ : $O\left(\frac{N}{2^j}\right)$ ops

$\vdots$

level $0$ $\{d_J(0), c_J(0)\}$ : $O(1)$ ops

$\Rightarrow$ Total cost

$O(N)$ ops.

Compare this to FFT which requires $O(N \log N)$ ops.