# Multidimensional Wavelet Representations

It is possible to construct wavelet representations and DWTs for multidimensional signals. In particular, we will be interested in 2-d DWTs for image analysis.

The basic scale space construction is completely analogous to the 1-d case.

We begin by specifying a scaling function $\phi(\underline{r})$ where $\underline{r} \in \mathbb{R}^n$ and generate a space

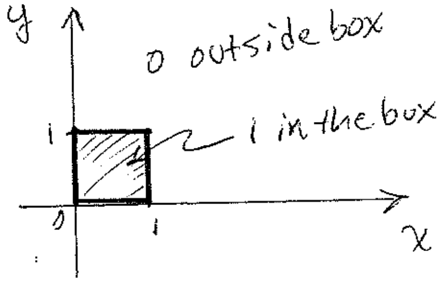
$$V_0 = \{ \phi(\underline{r} - \underline{k}) \}_{\underline{k} \in \mathbb{Z}^n}$$

n-tuples of integers

Ex. 2-d Haar scaling function

Let $\underline{r} = (x, y)$ point in the plane

$$\phi(x,y) = \begin{cases} 1, & 0 \le x < 1,\ 0 \le y < 1 \\ 0, & \text{o.w.} \end{cases}$$




This is a very natural approximation function for images since we normally associate each pixel value with the integral of the underlying (continuous) image intensity function over a small square region of space.

# 2-d MRA

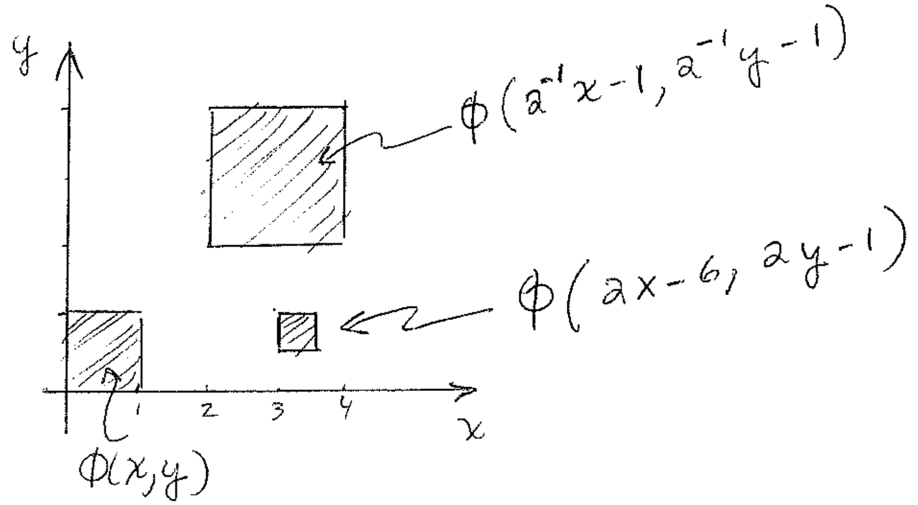
The 2-d multiresolution subspaces are generated by translates and dilates of the scaling function.

$$V_j = \{ 2^{j(n/2)} \phi(2^j \underline{r} - \underline{k}) \}_{k \in \mathbb{Z}^n}$$

And we have

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots L^2(\mathbb{R}^n)$$

Ex. Haar scaling functions

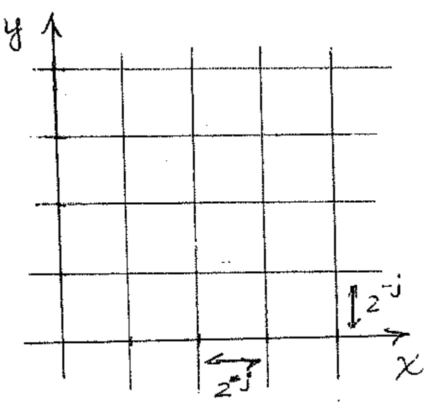


## 2-d Wavelet Subspaces

Difficulty:

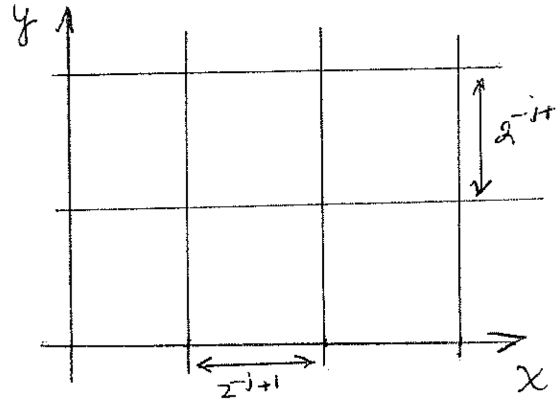
Going from scale $2^{-j}$ to $2^{-j+1}$ (equiv. resolution $2^{j}$ to $2^{j-1}$) involves a loss of information in the ratio of $2^n : 1$.

Ex. Haar Analysis




support of scaling functions at resolution $2^{j}$

$\Longrightarrow$



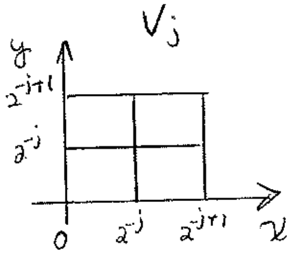

support of scaling functions at resolution $2^{j-1}$

At scale $2^{-j}$, each $2^{-j+1} \times 2^{-j+1}$ region of the plane is "analyzed" by 4 scaling functions. In contrast, at scale $2^{-j+1}$, only one scaling function analyzes the same region.

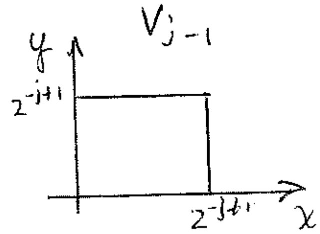
To account for the $2^n : 1$ loss of information between $V_j$ and $V_{j-1}$ it is necessary to have $2^n - 1$ wavelet functions (or equivalently $2^n - 1$ highpass filters) to carry this lost info.

Ex. Haar Analysis

To represent the difference between



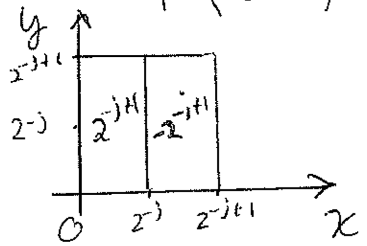

and




we need to represent the deviations of the $V_j$ piecewise constant approximation over the region $[0, 2^{-j+1}) \times [0, 2^{-j+1})$ from the $V_{j-1}$ constant approximation over the same region.

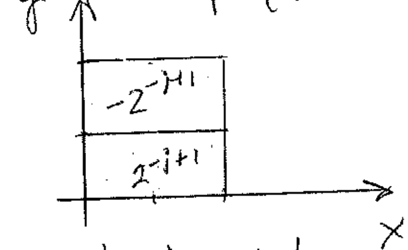
These deviations or "details" are represented by projecting the image onto the 2-d Haar wavelets:

$\psi^1(2^{-j+1}x, 2^{-j+1}y)$



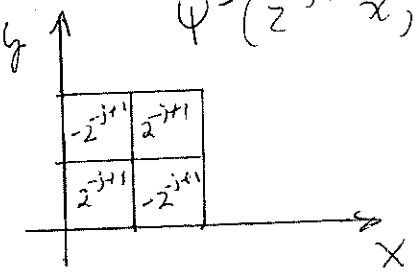
vertical

$\psi^2(2^{-j+1}x, 2^{-j+1}y)$



horizontal

$\psi^3(2^{-j+1}x, 2^{-j+1}y)$



diagonal

Note:

$\psi^1$ carries the horizontal deviation (i.e., senses vertical edges or details)

$\psi^2$ carries vertical deviation (i.e., senses horizontal edges)

$\psi^3$ carries diagonal deviation (i.e., sensitive to diagonal structure)

## 2-d Wavelet Subspaces

So, in 2-d we have 3 wavelet subspaces at each scale

$$W_j^1, \; W_j^2 \text{ and } W_j^3$$

$$V_{j+1} = V_j \oplus W_j^1 \oplus W_j^2 \oplus W_j^3$$

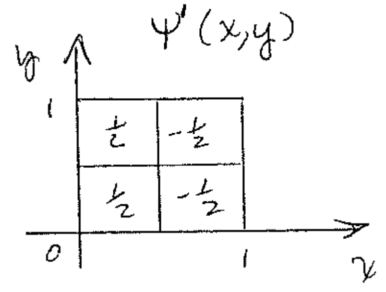
Ex. Let $\psi^{\ell}_{j,m,n} = \psi^{\ell}(2^j x - m, 2^j y - n)$ (Haar wavelets)

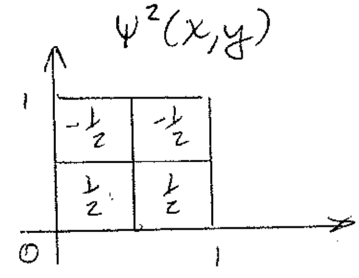
Verify that

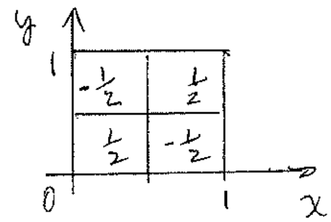
$$P_{V_{j+1}} f(x,y) = P_{V_j} f(x,y) + P_{W_j^1} f(x,y) + P_{W_j^2} f(x,y) + P_{W_j^3} f(x,y)$$

Hint: It suffices to show that each scaling function $\phi(2^{j+1}x + m, 2^{j+1}y + n)$ can be expressed as a linear combination of $\{\phi(2^j x + m, 2^j y + n)\}$ and $\{\psi^{\ell}(2^j x + m, 2^j y + n)\}_{\ell=1,2,3}$

Note that the prototype Haar wavelets




can all be represented as products of 1-d Haar wavelets and scaling functions

$$\psi^1(x,y) = \psi(x)\,\phi(y)$$

$$\psi^2(x,y) = \phi(x)\,\psi(y)$$

$$\psi^3(x,y) = \psi(x)\,\psi(y)$$

where $$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{o.w.} \end{cases}$$

A similar construction holds in general.

<u>Thm</u>: Let $\phi$ be a scaling function and $\psi$ be the corresponding wavelet generating an orthogonal MRA of $L^2(\mathbb{R})$. Define three wavelets:

$$\psi^1(x,y) = \psi(x)\,\phi(y) \qquad \psi^3(x,y) = \psi(x)\,\psi(y)$$

$$\psi^2(x,y) = \phi(x)\,\psi(y)$$

and for $l = 1, 2, 3$ let

$$\psi^l_{j,m,n} = 2^j \psi^l(2^j x - m,\ 2^j y - n).$$

Then
$$\{\psi^l_{j,m,n}\}_{m,n\in\mathbb{Z}^2} \text{ is an o.n. basis for } W_j^l$$

and

$$\{\psi^1_{j,m,n},\ \psi^2_{j,m,n},\ \psi^3_{j,m,n}\}_{j,m,n\in\mathbb{Z}^3}$$

is an o.n. basis for $L^2(\mathbb{R}^2)$.

## The 2-d DWT

Let $f(x,y)$ be a 2-d image, $x,y \in \mathbb{R}^2$, and let $\{\phi(x-m, y-n)\}_{m,n}$ be an o.n. basis for $V_0$ and

$\{\psi^{\ell}_{j,m,n}(x,y)\}_{m,n}$ be an o.n. basis for $W_j^{\ell}$, $\ell = 1, 2, 3$, $j \geq 0$.

Then we can write

$$f(x,y) = \sum_{m,n=-\infty}^{\infty} c_0(m,n)\, \phi(x-m, y-n) + \sum_{j \geq 0} \sum_{\ell=1}^{3} \sum_{m,n=-\infty}^{\infty} d_j^{\ell}(m,n)\, \psi^{\ell}_{j,m,n}(x,y)$$

where

$$c_0(m,n) = \langle f, \phi(x-m, y-n) \rangle$$

$$d_j^{\ell}(m,n) = \langle f, 2^j \psi(2^j x - m, 2^j y - n) \rangle$$

$\{d_j^{\ell}(m,n)\}_{m,n}$ = wavelet coefficients at scale $2^{-j}$ and orientation $\ell$.

## Computing the 2-d DWT

① Assume an initial set of scaling coefficients $\{c_J(m,n)\}_{m,m}$ representing an approximation $f_J = P_{V_J} f$ ($\approx f$ if $J$ is suff. large) to an image $f$ at scale $J$.

In practice, $\{c_J(m,n)\}$ are the pixel values of a digital image.

② The wavelet and scaling coefficients at coarser scales, $j < J$, are computed recursively using a <u>1-d</u> lowpass scaling filter $\{h(n)\}_n$ and a <u>1-d</u> highpass filter $\{h_1(n) = (-1)^n h(1-n)\}_n$.

(Exploiting separability of wavelet basis functions)

ⓐ $c_j(m',n') = \langle f, 2^j \phi(2^j x - m, 2^j y - n)\rangle$

$$= \sum_{m,n} h(m-2\ )\,h(n-2n')\,c_{j+1}(m',n')$$

ⓑ $$d_j^1(m',n') = \sum_{m',n'} h_1(m-2m')\,h(n-2n')\,c_{j+1}(m',n')$$

highpass filter rows (horizontal) — lowpass filter columns (vertical)

$$d_j^2(m',n') = \sum_{m',n'} \underbrace{h}_{\text{lowpass}}(m-2m')\,\underbrace{h_1}_{\text{highpass}}(n-2n')\,c_{j+1}(m',n')$$

$$d_j^3(m',n') = \sum_{m',n'} h_1(m-2m')\,h_1(n-2n')\,c_{j+1}(m',n')$$

highpass — highpass

<u>Note:</u> Decimation in both vertical and horizontal directions

# Organization and Display of 2d-DWT

Assume we begin with a $2^J \times 2^J$ digital image $f_J$ (i.e., $\{c_J(m,n)\}_{m,n=0}^{2^J-1}$).

Because of decimation at each stage, we have

$$\{d_j^\ell(m,n)\}_{m,n=0}^{2^j-1}$$

$2^j \times 2^j$ instead of $2^J \times 2^J$

$0 \le j < J$.

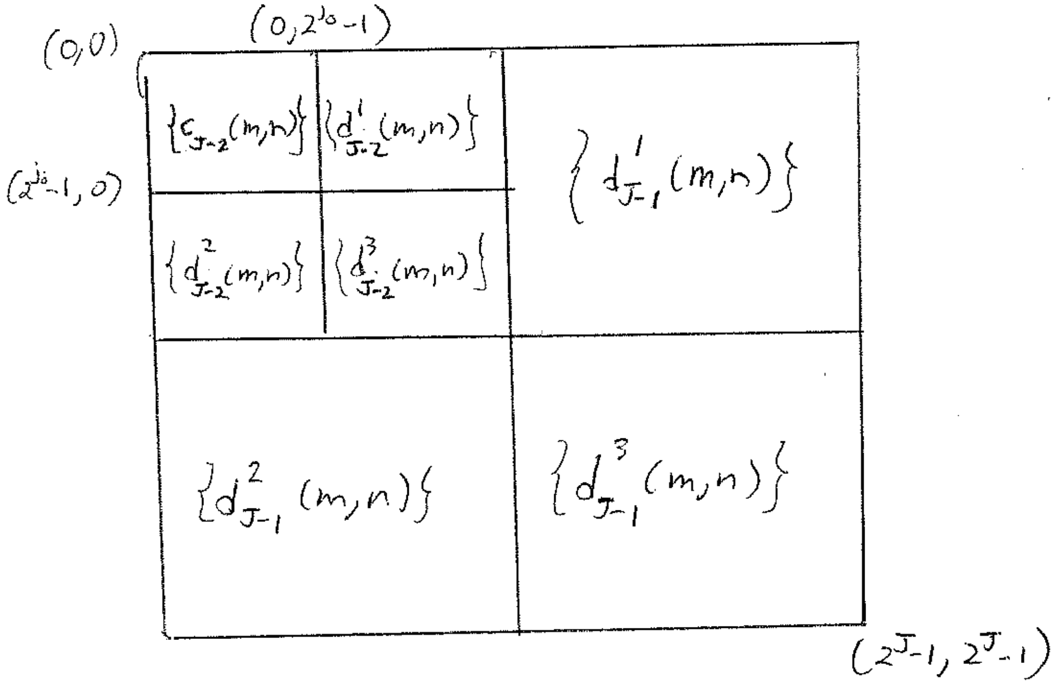
Note that the $j_0$-scale ($j_0 \le J$) DWT of $f_J$ produces

$$\{c_{j_0}(m,n)\}_{m,n=0}^{2^{j_0}-1} \text{ and } \{d_j^\ell(m,n)\}_{m,n=0}^{2^j-1}$$

$j = j_0, \ldots, J-1$

Exactly $2^J \times 2^J$ scaling and wavelet coefficients.

We can organize the coeff. into an image, e.g., $j_0 = 2$,




The "subimages" are collections of wavelet or scaling coefficients at a particular scale (and orientation in the wavelet case)

# DWT of an Image

Image

Scaling and Wavelet Coefficients