Time-Frequency Analysis

Linear Analysis

Decompose the signal $x(t)$ into elementary "time-frequency atoms" (e.g., Gabor functions):

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{S_x(u,f)}_{\text{"weights"}} \underbrace{g(t-u)e^{j2\pi f t}}_{\text{Gabor functions}} \, df \, du$$

The weights in this superposition reflect the distribution of signal energy in time and freq.

Quadratic Analysis

Goal is to find a time-freq energy density $\rho_x(t,f)$ satisfying

$$\int_{-\infty}^{\infty} \rho_x(u,f)\,du = |X(f)|^2$$

$$\int_{-\infty}^{\infty} \rho_x(t,\nu)\,d\nu = |x(t)|^2$$

Clearly, this requires that ρ_x is a quadratic function of $x(t)$.

Linear Time-frequency Analysis

The basic idea is to represent a signal x in terms of elementary "building-block" signals that are well concentrated in time and frequency. These elementary signals are called "time-frequency atoms".

Let $\{\phi_\gamma\}_{\gamma \in \Gamma}$ be a family of TF atoms. Here, the index γ may indicate both time- and frequency localization; e.g.,

$$\gamma = (t_0, f_0)$$

Let us assume that all atoms are normalized

$$\|\phi_\gamma\|_2 = \left(\int |\phi_\gamma(t)|^2 dt \right)^{\frac{1}{2}} = 1 .$$

Then we can define a linear time-frequency operator T as

$$Tx(\gamma) = \int_{-\infty}^{\infty} x(t)\, \phi_\gamma^*(t)\, dt$$

$$= \langle x, \phi_\gamma \rangle$$

Also, by Parsevol's Theorem

$$Tx(\gamma) = \int_{-\infty}^{\infty} X(f)\, \Phi^*(f)\, df$$

where X, Φ are Fourier transforms of x, ϕ.

So, if ϕ_γ is concentrated in a neighborhood about time t_0, then $Tx(\gamma)$ depends only on values of x in this neighborhood. Similarly, if Φ_γ is concentrated in a neighborhood about frequency f_0, then $Tx(\gamma)$ only depends of $X(f)$ in this neighborhood.

Ex.

Windowed Fourier Transform

$$\phi_\gamma(t) = e^{j2\pi f_o t} \underbrace{w(t-t_o)}_{\text{window function}}$$

Wavelet Transform

$$\phi_\gamma(t) = \frac{1}{\sqrt{s_o}} \psi\left(\frac{t-t_o}{s_o}\right)$$

$\gamma = (t_o, s_o)$, $\psi(t) =$ "mother" wavelet

scale instead of frequency

scale $s_o \propto \frac{1}{f_o}$

smaller scale

$\Longleftrightarrow$ higher frequency

Time-Frequency Concentration

The energy of ϕ_γ is centered at

$$t_\gamma = \int_{-\infty}^{\infty} t \, |\phi_\gamma(t)|^2 \, dt$$

in time, and in frequency at

$$f_\gamma = \int_{-\infty}^{\infty} f \, |\Phi_\gamma(f)|^2 \, df$$

The energy "spread" about (t_γ, f_γ) is measured by

$$\sigma_t^2(\gamma) = \int_{-\infty}^{\infty} (t - t_\gamma)^2 \, |\phi_\gamma(t)|^2 dt$$

$$\sigma_f^2(\gamma) = \int_{-\infty}^{\infty} (f - f_\gamma)^2 \, |\Phi_\gamma(f)|^2 df$$

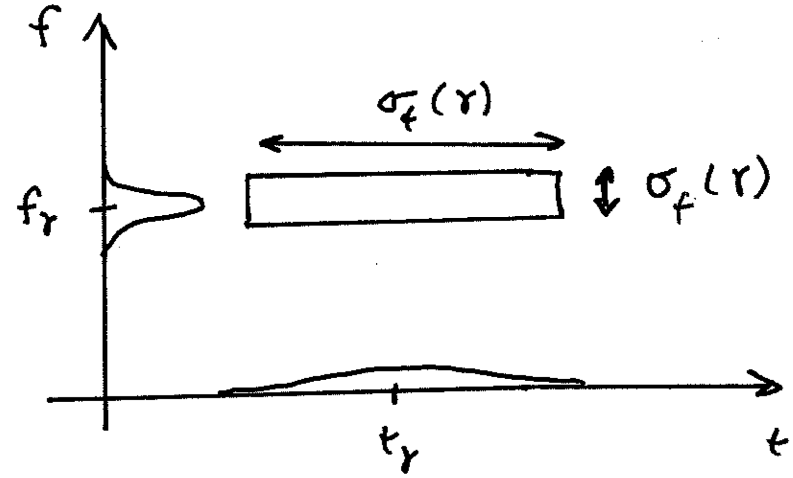
Heisenberg Uncertainty

$$\Longrightarrow \quad \sigma_t(\gamma)\, \sigma_f(\gamma) \geq$$

Time-Frequency Energy Density

$$\left| \langle x, \phi_\gamma \rangle \right|^2 = \left| \int_{-\infty}^{\infty} x(t)\, \phi_\gamma^*(t)\, dt \right|^2$$

measures the energy of x
in the neighborhood of (t_γ, f_γ).



$|Tx(\gamma)|^2$ is a time-frequency energy distribution function of x.

Short-Time Fourier Transform (STFT)

Replace global Fourier analysis

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

with a sequence of local analyses with respect to a moving observation window

$$Sx(t,f) = \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi f u} \underbrace{w(u-t)}_{\text{window about time } t.} du$$

$$= \int_{-\infty}^{\infty} X(\nu) \underbrace{W(\nu - f)}_{\text{window about frequency } f.} e^{-j2\pi(\nu - f)t}\, d\nu$$

Common windows

Gaussian $\Rightarrow$ Gabor analysis

rectangle, Hamming, Blackman

$w(t)$ assumed to be real, symmetric and $\int |w(t)|^2 dt = 1$.

Ex. $x(t) = e^{j2\pi f_0 t}$

$$S_x(t,f) = \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi f u}\, w(u-t)\,du$$

$$= \int_{-\infty}^{\infty} X \quad W(\nu-f)\, e^{-j2\pi(\nu-f)t}\, d\nu$$

Ex. $x(t) = \delta(t - t_0)$

Spectrogram

- The energy distribution resulting from the STFT is called the spectrogram

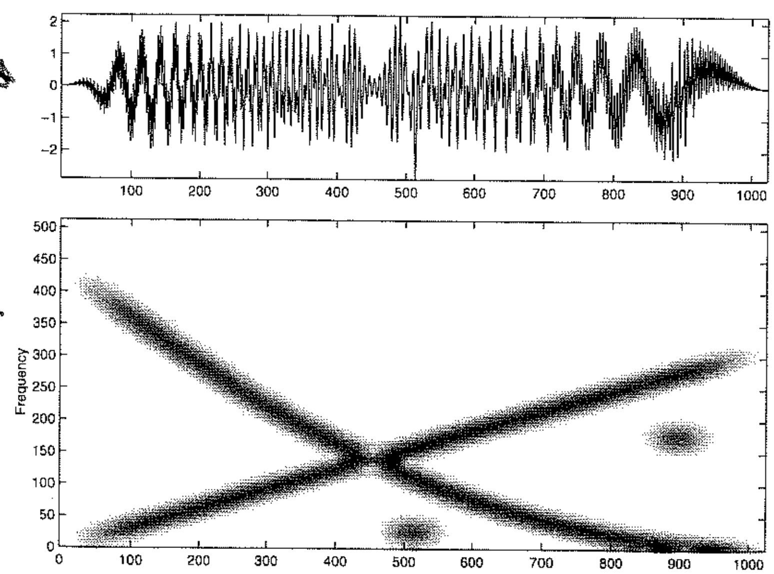
$$P_{S_x}(t,f) = |Sx(t,f)|^2$$

$$= \left| \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi f u}\, w(u-t)\, du \right|^2$$

- Ex. $w(u) = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-u^2/2\sigma^2}$, $\sigma = 50$

signal

spectrogram



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Time-frequency Resolution of Spectrogram

Because $w(u)$ is assumed to be real and symmetric, both the time and frequency spread about (t,f) is independent of (t,f):

$$\sigma_t^2 = \int_{-\infty}^{\infty} (t-u)^2 \left| w(t-u)\, e^{-j2\pi f u} \right|^2 du$$

$$= \int_{-\infty}^{\infty} u^2 \left| w(u) \right|^2 du, \quad \text{indep. of } t$$

$$\sigma_f^2 = \int_{-\infty}^{\infty} (f-\nu)^2 \left| W(f-\nu)\, e^{-j2\pi(\nu-f)t} \right|^2 d\nu$$

$$= \int_{-\infty}^{\infty} \nu^2 \left| W(\nu) \right|^2 d\nu, \quad \text{indep. of } f$$

$\Rightarrow$ time resolution σ_t and frequency resolution σ_f are independent of position in the TF plane.