Introduction to Time-Frequency Analysis

Now let's go back to a more basic question. Why frequency (spectral) analysis?

Let $x(t)$ be a deterministic, continuous-time signal. The Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{-j2\pi f t}\, df$$

show us that this signal can be represented with an (infinite) superposition of complex sinusoids. The spectral analysis problem aimed at determining the density of this superposition from a finite number of noisy measurements of x.

But, perhaps this is the wrong question/problem in the first place. While it is true that we generally have a Fourier representation, the individual complex sinusoid components of the superposition lack a solid physical interpretation.

Complex sinusoids are:

1. everlasting
2. completely nonlocal

Most real-world signals, on the other hand, are:

1. essentially time-limited
2. localized in time

Time-frequency analysis takes a slightly different approach. It aims to describe how the frequency content (spectral density) varies in time. This raises a myriad of questions and problems — frequency <u>means</u> variation in time; how can frequencies vary over time!! It should sound like a slightly circular problem (because in many ways it is)

A starting point for time-frequency analysis is the following question: Can a signal be (approximately) time and band limited?

Heisenberg's Uncertainty Principle

Does the notion of a time-frequency energy density make sense?

Such a density should measure the signal energy at different points in time and frequency.

To measure energy at

time t_0 : $E_{t_0} = \left(\int x(t)\, \delta(t-t_0)\, dt \right)^2$

frequency f_0 : $E_{f_0} = \left(\int x(t)\, e^{-j2\pi f_0 t}\, dt \right)^2$

The function

$\delta(t-t_0)$ is localized in time

$e^{-j2\pi f_0 t}$ is localized in frequency

Thus, to measure the energy at time t_0 and frequency f_0, we require a function $g_{t_0,f_0}(t)$ that is localized in both time and frequency (about (t_0,f_0)):

$$E_{t_0,f_0} = \left(\int x(t)\, g_{t_0,f_0}(t)\, dt \right)^2$$

Does such a g_{t_0,f_0} exist?

Let's see what we can do.

Let us postulate a function g that is bandlimited to $f \in [-B/2, B/2]$ and timelimited to $t \in [-\frac{T}{2}, \frac{T}{2}]$. Any non-zero function satisfying these properties would have to satisfy the relation

$$g(t) = \int_{-B/2}^{B/2} G(f)\, e^{-j 2\pi f t}\, df = 0$$

$$\text{for} \quad |t| > \frac{T}{2}$$

Since g is of bounded (time) duration, the same must be true for its n^{th} derivative: *

$$\frac{d^n g(t)}{dt^n} = \int_{-B/2}^{B/2} (j2\pi f)^n G(f)\, e^{j2\pi f t}\, df = 0$$

$$|t| > \frac{T}{2}, \qquad n \geq 0.$$

* Bandlimited $\Rightarrow$ analytic
(i.e., very smooth and differentiable)

The value of g at a point $s \in [-\frac{T}{2}, \frac{T}{2}]$ can be written as

$$g(s) = \int_{-\frac{B}{2}}^{\frac{B}{2}} G(f) e^{j2\pi f(s-t)} e^{j2\pi f t} \, df$$

$$|t| > \frac{T}{2}$$

By replacing $e^{j2\pi f(s-t)}$ with its power series

$$e^{j2\pi f(s-t)} = \sum_{n=0}^{\infty} \frac{(j2\pi(s-t))^n}{n!} f^n$$

we have

$$g(s) = \sum_{n=0}^{\infty} \frac{(s-t)^n}{n!} \underbrace{\int_{-\frac{B}{2}}^{\frac{B}{2}} (j2\pi f)^n G(f) e^{j2\pi f t} \, df}_{= \frac{d^n \, (t)}{dt^n} = 0 \text{ since } |t| > \frac{T}{2}}$$

Thus, for all $|s| < \frac{T}{2}$ we have $g(s) = 0$!!

This contradicts our initial assumption that the signal is non-zero in the interval $[-\frac{T}{2}, \frac{T}{2}]$.

Conclusion: The only signal that is exactly time and band limited (to any intervals) is the trivial function $g \equiv 0$.

So, we must relax our strict constraint of finite (compact) support in both time and frequency.

We know that a short pulse in time extends over a broad frequency range. Vice-versa, the narrower the band of a filter, the longer its impulse response. At the extremes

$$\delta(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow \delta(f)$$

A less extreme case is seen with Gaussian functions.

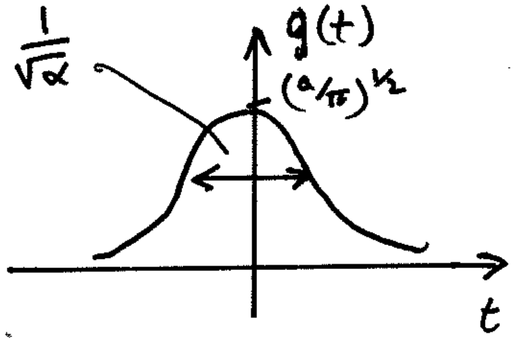
Let

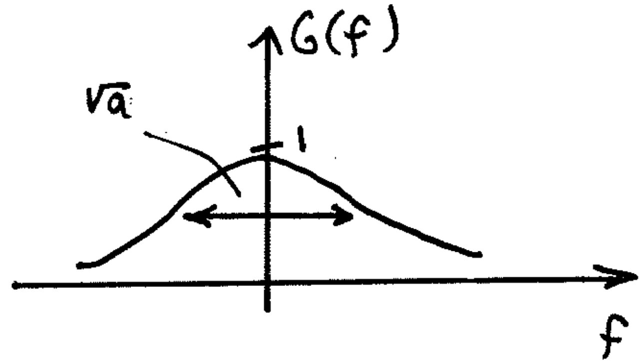
$$g(t) = \left(\frac{a}{\pi}\right)^{1/2} e^{-\alpha t^2}$$

Then

$$G(f) = \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha t^2} e^{-j2\pi f t}\, dt$$

$$= e^{-\pi^2 f^2/\alpha}$$





The "width" or essential support of g in time is $\frac{1}{\sqrt{\alpha}}$ and in frequency the essential support is $\sqrt{\alpha}$.

$\alpha \uparrow \Rightarrow$ time support frequency support

$\alpha \downarrow \Rightarrow$ time support frequency support

What happens as $\alpha \to 0$ or $\alpha \to \infty$?

We are now in a position to pose the following question.

How time and band limited can a function g be? We already know that there must be a limit (g can't be time and band limited).

The answer lies in Heisenberg's famous <u>Uncertainty Principle</u>.

Recall: We cannot know the position and momentum of a particle simultaneously. The more

Let us define the time and frequency concentrations of g about a point (t_0, f_0) as

$$\sigma_t^2 = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty} (t-t_0)^2 |g(t)|^2 dt$$

$$\sigma_f^2 = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty} (f-f_0)^2 |G(f)|^2 df$$

<u>Theorem</u>: If $g \in L_2$ $\left(\|g\|^2 = \int |g(t)|^2 dt < \infty \right)$,

then

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{(4\pi)^2}$$

for every t_0, f_0.

<u>Proof</u>: We will assume that

$$\lim_{|t|\to\infty} t\, g^2(t) = 0,$$ implying

that $g(t)$ decays sufficiently fast.
However, the Theorem holds for
any $g \in L_2$ (but this requires a
bit more work to prove).

First note that if g is concentrated
about (t_0, f_0), then $h(t) \equiv e^{-j2\pi f_0 t} g(t+t_0)$
is concentrated about $(0,0)$. Therefore,
without loss of generality, we can
assume that $(t_0, f_0) = (0,0)$. Let's also
assume that $\| \ \|^2 = 1$, for convenience.

By definition

$$\sigma_t^2 \sigma_f^2 = \int_{-\infty}^{\infty} |t\, g(t)|^2 dt \int_{-\infty}^{\infty} |f\, G(f)|^2 df$$

Next, since

$$g'(t) \longleftrightarrow j2\pi f\, G(f)$$

we can apply Parsevals Theorem to get

$$\int_{-\infty}^{\infty} |f G(f)|^2 df = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |g'(t)|^2 dt$$

Therefore,

$$\sigma_t^2 \sigma_f^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |t g(t)|^2 dt \int_{-\infty}^{\infty} |g'(t)|^2 dt$$

Now recall the Schwarz Inequality:

$$\left| \int x(t) y^*(t)\, dt \right|^2 \leq \int |x(t)|^2 dt \cdot \int |y(t)|^2 dt$$

Applying the Schwartz inequality in our case gives

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{4\pi^2}\left(\int_{-\infty}^{\infty} t\, g(t)\, g'(t)\, dt\right)^2$$

Observe that

$$t\, g(t)\, g'(t) = \frac{t}{2}\left(g^2(t)\right)'$$

Now use integration by parts

$$\left(\int_{-\infty}^{\infty} u\, dv = uv\Big|_{-\infty}^{\infty} - \int v\, du,\quad u = t,\ v = \frac{g^2(t)}{2}\right)$$

$$\int_{-\infty}^{\infty} \frac{t}{2}\left(g^2(t)\right)'\, dt = \underbrace{t\,\frac{g^2(t)}{2}\Big|_{-\infty}^{\infty}}_{\substack{=0 \text{ since} \\ \lim_{|t|\to\infty} t\, g^2(t) = 0}} - \frac{1}{2}\int_{-\infty}^{\infty} g^2(t)\, dt$$

$\Rightarrow$

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{16\pi^2}\Big(\underbrace{\int_{-\infty}^{\infty} g^2(t)\, dt}_{\|g\|^2 = 1}\Big)^2 = \frac{1}{(4\pi)^2}$$

$\blacksquare$

Summary: Let x be a CT signal

Energy at time t_0:

$$E_{t_0} = \left(\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt \right)^2$$

$$= \langle x, \delta(\cdot - t_0) \rangle$$

Energy at frequency f_0:

$$E_{f_0} = \left(\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f_0 t}\,dt \right)^2$$

$$= \langle x, e^{-j2\pi f_0 (\cdot)} \rangle$$

Energy at time t_0 and frequency f_0:

$$E_{t_0,f_0} = \left(\int_{-\infty}^{\infty} x(t)\, g_{t_0,f_0}(t)\,dt \right)^2 = \langle x, g_{t_0,f_0} \rangle$$

g_{t_0,f_0} cannot be totally local in both time and frequency

Heisenberg: $\sigma_t^2 \sigma_f^2 \geq \frac{1}{(4\pi)^2}$