Parametric Methods for Spectral Estimation

Non-parametric methods:

Periodogram
Blackman-Tukey
Periodogram Averaging

<u>Key</u>: No assumptions on signal under study, except stationarity

Parametric methods:

* model-based *

If assumed model is a close approximation to reality, then parametric methods outperform nonparametric ones.

1. assume signal was generated according to a parameterized model.
2. estimate model parameters.
3. power spectral density is then derived from estimated model.

Signals with Rational Spectra

A rational power spectral density is a ratio of two polynomials in $e^{-j2\pi f}$:

$$\Gamma(f) = \frac{\sum_{k=-q}^{q} \mu_k e^{-j2\pi f k}}{\sum_{k=-p}^{p} \rho_k e^{-j2\pi f k}}$$

Where $\mu_{-k} = \mu_k^*$ and $\rho_{-k} = \rho_k^*$.

The Weierstrauss Theorem asserts that any continuous power spectral density can be approximated arbitrarily closely by a rational function, provided the degrees p and q are sufficiently large.

Because $\Gamma(f) \geq 0$, the rational power spectral density can be factored as

$$\Gamma(f) = \frac{|B(f)|^2}{|A(f)|^2}\sigma^2$$

Where σ^2 is a positive scalar, and $A(f)$ and $B(f)$ are the polynomials

$$A(f) = 1 + a_1 e^{-j2\pi f} + \cdots + a_n e^{-j2\pi f p}$$

$$B(f) = 1 + b_1 e^{-j2\pi f} + \cdots + b_m e^{-j2\pi f q}$$

Alternatively, we can express the power spectral density in the z-domain.

$$\Gamma(z) = \frac{\sum_{k=-q}^{q} \mu_k z^{-k}}{\sum_{k=-p}^{p} \rho_k z^{-k}}$$

or using

$$A(z) = 1 + a_1 z^{-1} + \dots + a_n z^{-p}$$
$$B(z) = 1 + b_1 z^{-1} + \dots + b_m z^{-q}$$

$$\Gamma(z) = \sigma^2 \frac{B(z)\, B^*(1/z^*)}{A(z)\, A^*(1/z^*)}$$

This is called the "Spectral Factorization"

Note:

$$A^*(1/z^*) = \left(A\left(\tfrac{1}{z^*}\right)\right)^*$$
$$= \left(1 + a_1 z^* + \dots + a_n z^{*\,p}\right)^*$$
$$= 1 + a_1^* z + \dots + a_n^* z^p$$

Generative Model

Recall:

$$\{w(n)\} \longrightarrow \boxed{H(f)} \longrightarrow \{x(n)\}$$

white noise	LTI system	stationary output

$$r_{ww}(m) = \sigma^2 \delta(m)$$

$$\Gamma_{ww}(f) = \sigma^2$$

$$\Gamma_{xx}(f) = \sigma^2 \left| H(f) \right|^2$$

In particular, if the (temporal) dynamics of the LTI system H are governed by the difference equation

$$x(n) = -\sum_{k=1}^{p} a_l\, x(n-k) + \sum_{k=0}^{q} b_k\, w(n-k)$$

Then

$$H(f) = \frac{B(f)}{A(f)}$$

and

$$\Gamma_{xx}(f) = \sigma^2 \frac{|B(f)|^2}{|A(f)|^2}$$

a rational power spectral density!

From

$$\Gamma_{xx}(f) = \sigma^2 \frac{|B(f)|^2}{|A(f)|^2} \qquad (*)$$

we see that the spectral estimation problem can be reduced to a problem of <u>signal modeling</u>. Moreover, the continuous function Γ_{xx} is actually only dependent on $p+q+1$ parameters

$$\{a_1, \ldots, a_p\}$$
$$\{b_1, \ldots, b_q\} \text{ and } \sigma^2.$$

This means that every Γ_{xx} of the form $(*)$ can be associated with a vector in $(p+q+1)$ dimensional space.

Does every unique vector produce a unique Γ_{xx}?

Three Signal Models:

ARMA :

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k w(n-k)$$

AR :

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + w(n)$$

MA :

$$x(n) = \sum_{k=0}^{q} b_k w(n-k)$$

Spectrum estimation boils down to estimating the parameters from a finite record of data, $\{\hat{a}_1, \ldots, \hat{a}_p, \hat{b}_1, \ldots, \hat{b}_q, \sigma^2\}$ and then plugging these in to form

$$\hat{\Gamma}(f) = \hat{\sigma}^2 \frac{|\hat{B}(f)|^2}{|\hat{A}(f)|^2}$$

Covariance Function and Model Parameters

- To obtain an expression for γ (γ_{xx}) in terms of $\{a_i\}$, $\{b_k\}$, σ^2, first write

$$x(n) + \sum_{i=1}^{p} a_i \, x(n-i) = \sum_{j=0}^{q} b_j \, w(n-j)$$

Multiplying by $x^*(n-k)$ and taking expectations yields

- $$\gamma(k) + \sum_{i=1}^{p} a_i \, \gamma(k-i) = \sum_{j=0}^{q} b_j \, E\left[w(n-j)\, x^*(n-k)\right]$$

To evaluate $E\left[w(n-j)\, x^*(n-k)\right]$ note that

$$x(n) = \sum_{k=0}^{\infty} h(k)\, w(n-k)$$

where $\{h(k)\}$ is the impulse response of $H(f) = B(f)/A(f)$.

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Thus

$$E\left[w(n-j)x^*(n-k)\right] = E\left[w(n-j) \sum_{=0}^{\infty} h^*(\ell)\, w^*(n-k-\ell)\right]$$

$$= \sigma^2 h^*(j-k)$$

and we have

$$\boxed{\gamma(k) + \sum_{i=1}^{p} a_i \gamma(k-i) = \sigma^2 \sum_{j=0}^{q} b_j h^*(j-k)} \qquad \#$$

In general, $h(k)$ is a nonlinear function of $\{a_i\}$ and $\{b_j\}$.

Note that if the $\{h(k)\}$ were not present above, then the covariance is linearly related to $\{a_i\}$ and $\{b_j\}$. In that case, estimates $\{\hat{a}_i\}$ and $\{\hat{b}_j\}$ can be found using our estimator $\hat{\gamma}(k)$ and solving a system of <u>linear</u> equations.

AR Model for Spectral Estimation

If we restrict ourselves to the AR(-only) model, then the covariance equations simplify to

$$\gamma(k) + \sum_{i=1}^{p} a_i \gamma(k-i) = \sigma^2 \mathbb{1}(k \le 0)$$

This system of equations is <u>linear</u> in $\{a_i\}$. In particular, for $k = 0, \ldots, n$ we have the following system of equations:

$$\begin{bmatrix} \gamma(0) & \gamma(-1) & \cdots & \gamma(-n) \\ \gamma(1) & \gamma(0) & & \vdots \\ \vdots & & \ddots & \gamma(-1) \\ \gamma(n) & \cdots & & \gamma(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using all but the first row (above) we have the system

$$\underbrace{\begin{bmatrix} r(1) \\ \vdots \\ r(n) \end{bmatrix}}_{\underline{r}} + \underbrace{\begin{bmatrix} r(0) & \cdots & r(-n+1) \\ \vdots & \ddots & \vdots \\ r(n-1) & \cdots & r(0) \end{bmatrix}}_{\underline{R}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\underline{a}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The solution is

$$\underline{a} = -\underline{R}^{-1}\underline{r}$$

Yule-Walker Method of Spectral Estimation

Replace $\{r(n)\}$ by $\{\hat{r}(n)\}$ (as defined on page 49) to form $\hat{\underline{r}}$, $\hat{\underline{R}}$

$$\Longrightarrow \hat{\underline{a}} = -\hat{\underline{R}}^{-1}\hat{\underline{r}}$$

$$\Longrightarrow \hat{\Gamma}(f) = \frac{\sigma^2}{|\hat{A}(f)|^2}$$

MA Model for Spectral Estimation

In the MA case,

$$\Gamma(f) = \sigma^2 \left| B(f) \right|^2 .$$

Furthermore, since the data are suppose to obey the model

$$w(n) \longrightarrow \underset{\substack{q\text{-length} \\ \text{FIR filter}}}{\boxed{B}} \longrightarrow x(n)$$

we have

$$\gamma(k) = 0 \quad , \quad |k| > q .$$

Thus, a natural spectral estimator in this case is simply

$$\hat{\Gamma}(f) = \sum_{k=-q}^{q} \hat{\gamma}(k)\, e^{-j2\pi f k}$$

where, again, $\hat{\gamma}(k)$ is defined on page 49. Note that this is simply the BT estimator with a rect window of length $2q+1$.

ARMA Model for Spectral Estimation

Recall that the covariance equations

$$\gamma(k) + \sum_{i=1}^{p} a_i \gamma(k-i) = \sigma^2 \sum_{j=0}^{q} b_j h^*(j-k)$$

are complicated, nonlinear functions of $\{a_i\}$ and $\{b_i\}$. Therefore, rather than attempting to directly solve for the ARMA parameters, we will take a two-step approach.

Note that since $h(n) = 0$ for $n<0$, the covariance equations for $\underline{k>q}$ simplify to

$$\gamma(k) + \sum_{i=1}^{p} a_i \gamma(k-i) = 0$$

Based on this observation, we can now state our two-step procedure.

Step 1:

Use equations

$$\hat{\gamma}(k) + \sum_{i=1}^{p} a_i \hat{\gamma}(k-i) = 0 \quad \text{for } k > q$$

and solve Yule-Walker equations

$$\underline{\hat{a}} = -\underline{\hat{R}}^{-1} \underline{\hat{r}}$$

$$\Rightarrow \quad \hat{P}_{AR}(f) = \frac{1}{|\hat{A}(f)|^2}$$

Step 2: Filter data $\{x(n)\}$ by FIR filter

$$g(n) = \begin{cases} \hat{a}_n, & 0 \le n \le p \\ 0, & \text{otherwise} \end{cases}$$

Note:

$$\begin{aligned} g(n) * x(n) &= G(f)\,X(f) \\ &= \hat{A}(f) \frac{B(f)}{A(f)} W(f) \\ &\approx B(f)\,W(f) \leftarrow \text{MA process!} \end{aligned}$$

Using the filtered sequence

$$y(n) = g(n) * x(n)$$

for an MA spectral estimate

$$\hat{\Gamma}_{MA}(f) = \sum_{k=-q}^{q} \hat{\gamma}_{yy}(k) e^{-j2\pi fk}$$

Now combine $\hat{\Gamma}_{AR}$ and $\hat{\Gamma}_{MA}$ to obtain the final ARMA-based estimate

$$\hat{\Gamma}_{ARMA}(f) = \frac{\hat{\Gamma}_{MA}(f)}{\hat{\Gamma}_{AR}(f)}$$

Ex.

$$x(n) = \sum_{i=1}^{4} A_i e^{j(2\pi f_i n + \phi_i)} + w(n)$$

$A_i = 1, \quad i = 1, 2, 3, 4$

$\phi_i \sim \text{Uniform}(0, 2\pi)$, independent

$\{w(n)\} \sim$ zero-mean, white noise

$f_1 = -0.222, \quad f_2 = -0.166, \quad f_3 = 0.10, \quad f_4 = 0.1222$

$n = 0, \ldots, 1023$