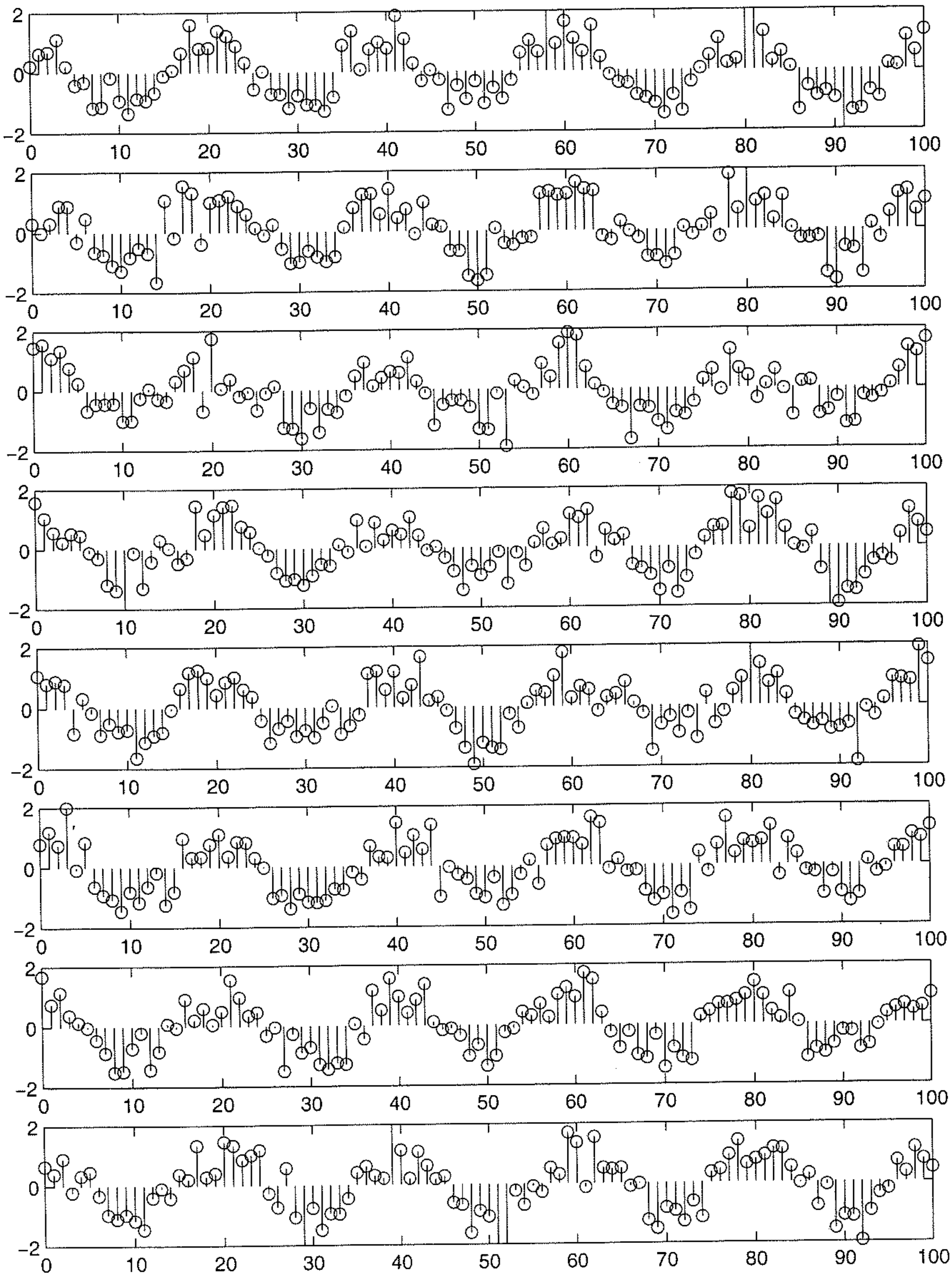
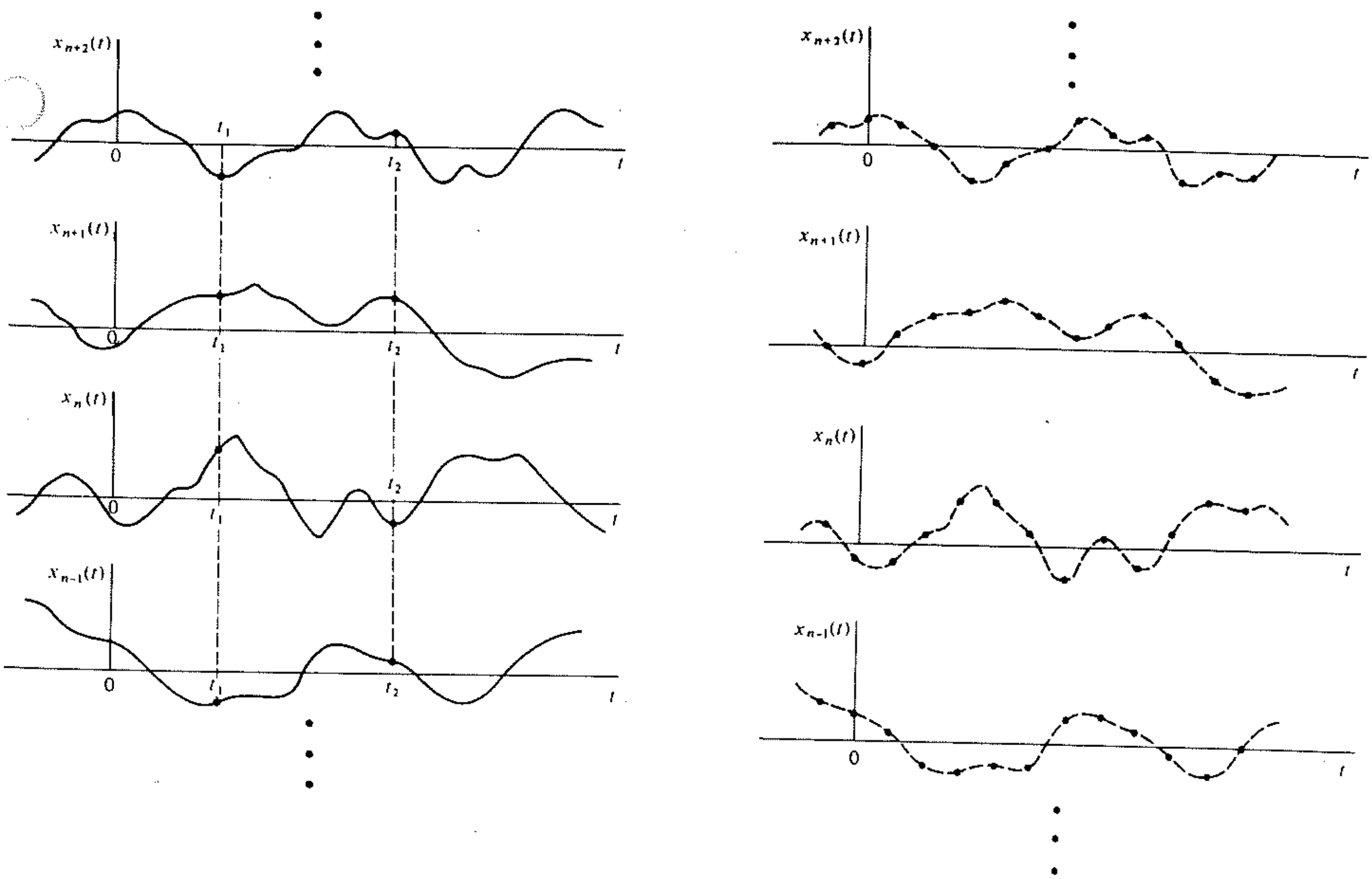
Discrete-Time Random Signals

Classic Example: Sinusoid + Noise

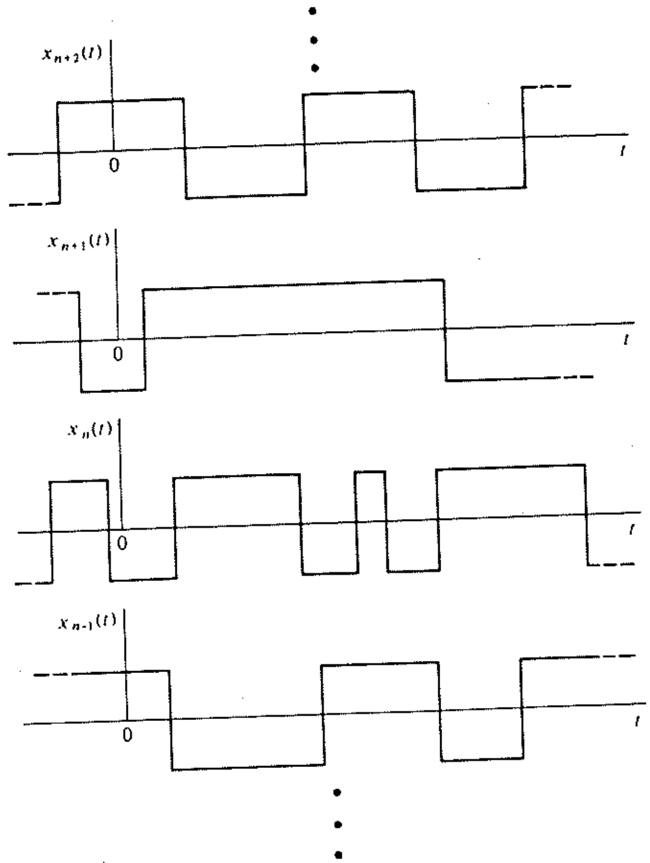


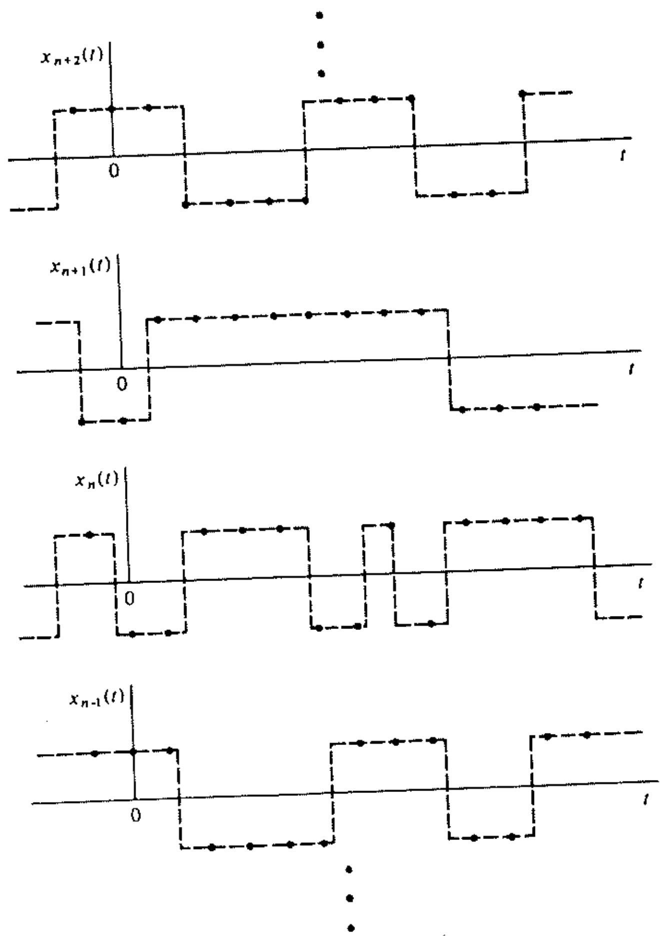
Definition : A random process is a family or ensemble of signals corresponding to every possible outcome of a certain signal measurement or experiment. Each signal in the ensemble is called a "realization" of the process.

Ex.



Ex. Random Binary Process



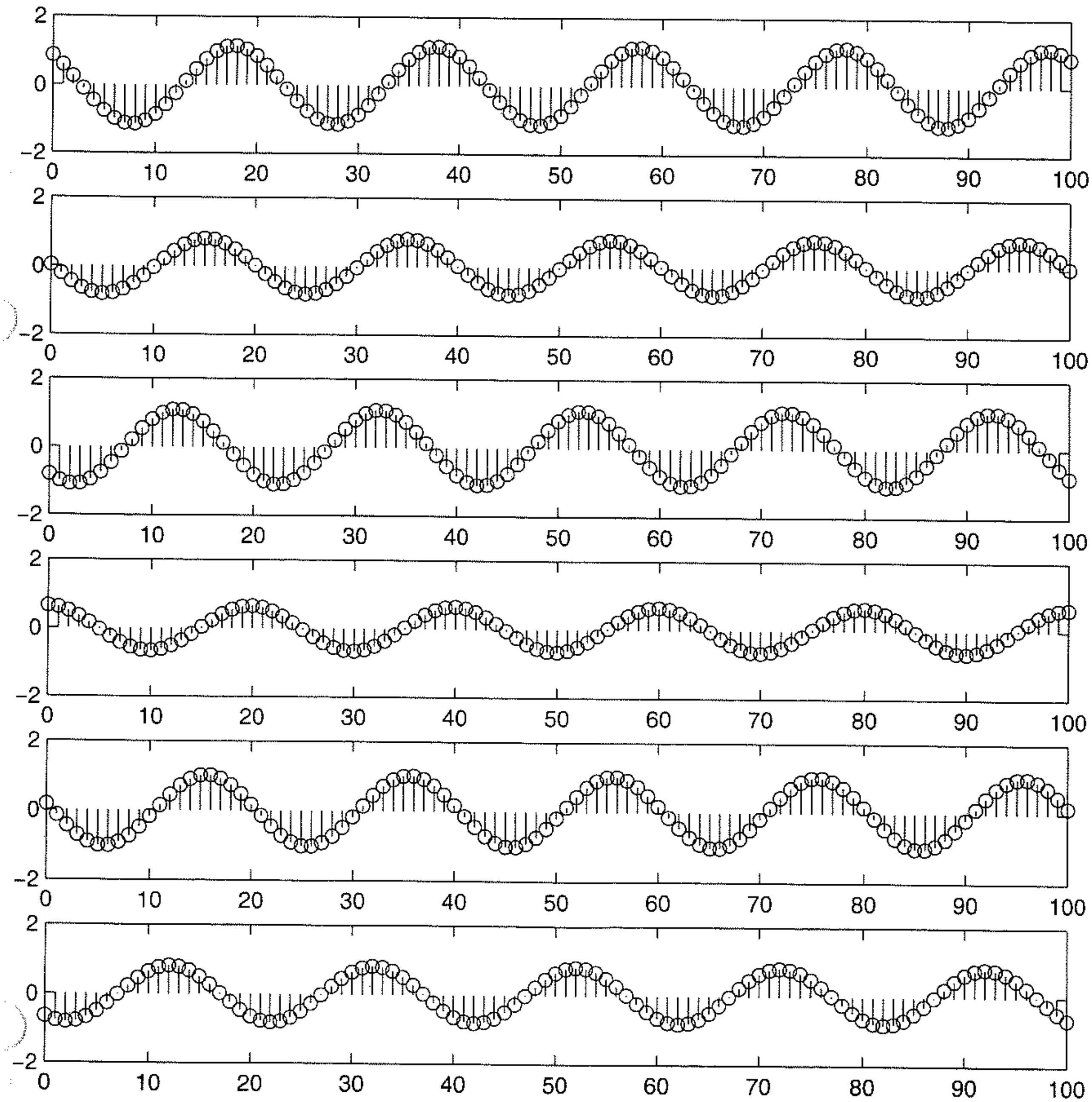


Ex. Random Sinusoidal Process

$$x[n] = A \cos(\omega_0 n + \phi)$$

A, ω_0, ϕ may all be random variables

random amplitude & phase



Definition: The <u>mean</u> of a random process is the average of all realizations of the process.

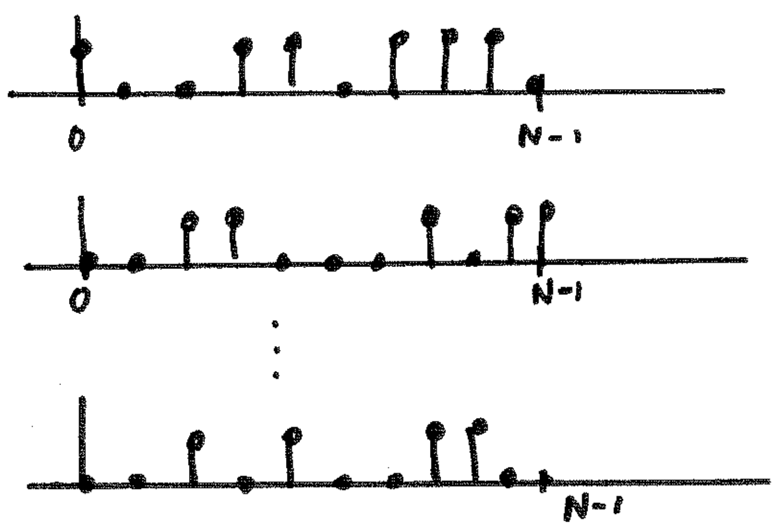
<u>Ex.</u> N-pt binary random sequence

$$x[n] = \begin{cases} 1, & \text{with prob } p \\ 0, & \text{with prob } (1-p) \end{cases} \qquad n = 0, \ldots, N-1$$

<u>**independent**</u>

$$\Pr(x[n]=1) = p \qquad \Pr(x[n]=0) = 1-p$$

Realizations:



What is $\text{mean}(x[n])$?

The mean is also known as the expectation, and is denoted by

$$m_x[n] = E\big[x[n]\big]$$

expectation operator

$\Rightarrow$ take average of all possible realizations of $x[n]$

First-order Stationary Processes

A process is 1st-order stationary if

$$m_x[n] = m_x \text{ (a constant independent of } n\text{)}$$

Ex. N-pt binary process revisited

First-order Stationarity and LTI Systems

$$X[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

1st-order stationary process

LTI system

output

LTI system is deterministic, but input is random.

What about output?

$$m_y[n] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right]$$

$$=$$

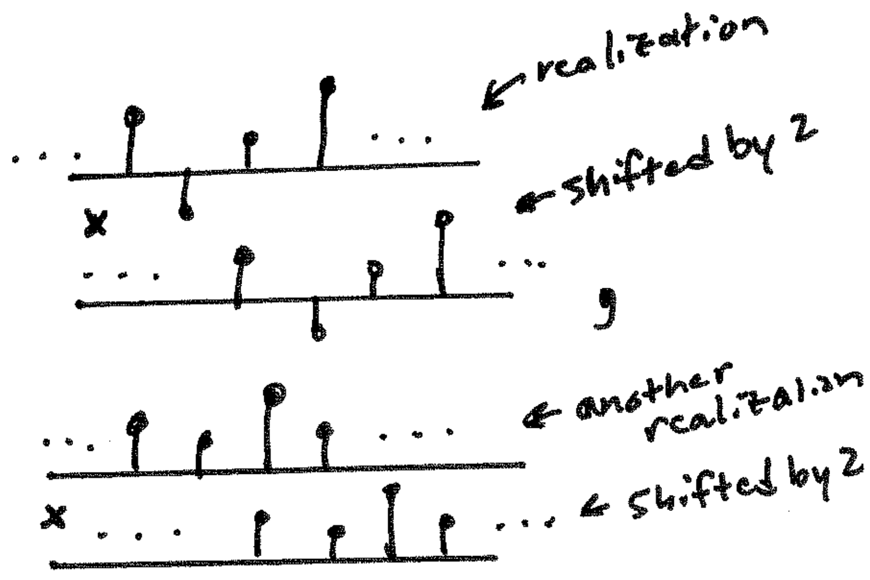
<u>Definition</u>: The <u>autocorrelation function</u> of a random process is the average product of a signal realization with a time-shifted version of itself.

$$\underbrace{R_{xx}}_{\text{autocorrelation function}}[n, n+m] = \underbrace{E}_{\text{Average over all possible realizations}}\left[\, x[n]\, x[n+m] \,\right]$$

<u>Ex.</u>

$R_{xx}[n, n+2] =$

Average $\{$



$\vdots \quad \}$

__Ex.__ Random Binary Process

Assume $\Pr\left(x[n]=1\right) = \Pr\left(x[n]=-1\right) = \frac{1}{2}$

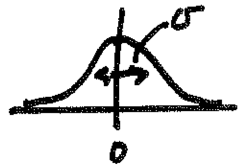
$$\Longrightarrow \quad m_x[n] =$$

$$R_{xx}[n, n+m] = E\left[x[n]\, x[n+m]\right]$$

Ex. Gaussian White Noise (GWN)

"white" $\Rightarrow$ $x[n]$ are independent and zero-mean

$$\Pr\left[a \le x[n] \le b\right] = \int_a^b \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}}_{\substack{\text{Gaussian density}\\ \text{function}\\ \sigma^2 = \text{variance}}} dx$$



$$m_x = E\left[x[n]\right] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2} dx = 0$$

$$R_{xx}[n, n+m] = E\left[x[n]\, x[n+m]\right]$$

$$=$$

Second-order Stationary Processes

A random process is 2nd-order stationary if

$$R_{xx}[n, n+m] \equiv R_{xx}[m]$$

That is, the autocorrelation function only depends on m, the shift, and not on n.

Ex. Binary Process

Ex. GWN

Stationary Inputs & LTI Systems

$$x[n] \longrightarrow \boxed{\;H\;} \longrightarrow y[n]$$

2nd order stationary process

LTI

$$m_y[n] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, E\big[x[n-k]\big] = m_x \sum_{k=-\infty}^{\infty} h[k]$$

$$R_{yy}[n, n+m] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \cdot \sum_{r=-\infty}^{\infty} h[r]\, x[n+m-r]\right]$$

$$=$$

$$\Rightarrow R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} R_{xx}[m-\ell] \cdot c_{hh}[\ell]$$

where $\quad c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]\, h[\ell+k]$

Ex. GWN into an LTI System

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

GNN

$$m_y = m_x \sum_{k=-\infty}^{\infty} h[k] = 0$$

$$R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \overset{\sigma^2 \delta[m-\ell]}{\cancel{R_{xx}[m-\ell]}} \cdot \sum_{k=-\infty}^{\infty} h[k]\,h[\ell+k]$$

$$= \sum_{\ell=-\infty}^{\infty} \sigma^2 \delta[m-\ell] \cdot \sum_{k=-\infty}^{\infty} h[k]\,h[\ell+k]$$

$$= \sum_{k=-\infty}^{\infty} \sigma^2 h[k]\,h[m+k]$$

Frequency Domain Analysis

(assume 2nd-order stationarity)

Power Spectrum

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}[m]\, e^{-j\omega m} \qquad ①$$

$$R_{xx}[m] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega m}\, d\omega \qquad ②$$

① & ② are called the Wiener-Khinchin Relations

Frequency Domain Analysis of LTI Systems

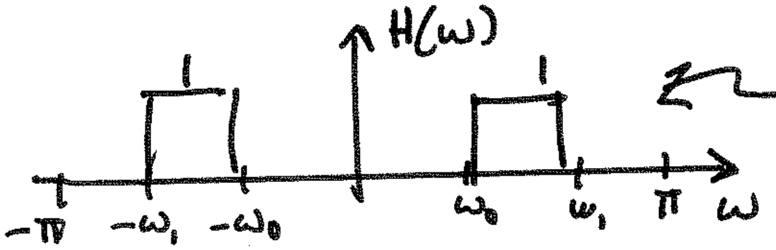
$$R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} R_{xx}[m-\ell]\, C_{hh}[\ell]$$

$$\Longrightarrow \quad S_{yy}(\omega) = |H(\omega)|^2 \cdot S_{xx}(\omega)$$

Power spectrum of $X[n]$ is "shaped" by $|H(\omega)|^2$

Ex.

$$H(\omega) = \begin{cases} 1, & \omega_0 \le |\omega| \le \omega_1 \\ 0, & \text{otherwise} \end{cases}$$



$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

In particular, the average output power is

$$E[Y^2[n]] = R_{YY}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_1}^{-\omega_0} S_{XX}(\omega)\, d\omega + \frac{1}{2\pi}\int_{\omega_0}^{\omega_1} S_{XX}(\omega)\, d\omega$$

This shows that we can interpret the area under $S_{XX}(\omega)$ for $\omega_0 \le |\omega| \le \omega_1$ as the average input power in that frequency band. Thus, $S_{XX}(\omega)$ can be viewed as a density function for power in the spectral (freq) domain.

Applications

Ex. $\boxed{\text{Noise Removal}}$

Suppose that we make noisy measurements of a random signal $s[n]$ in GWN $w[n]$:

$$x[n] = s[n] + w[n]$$

signal:

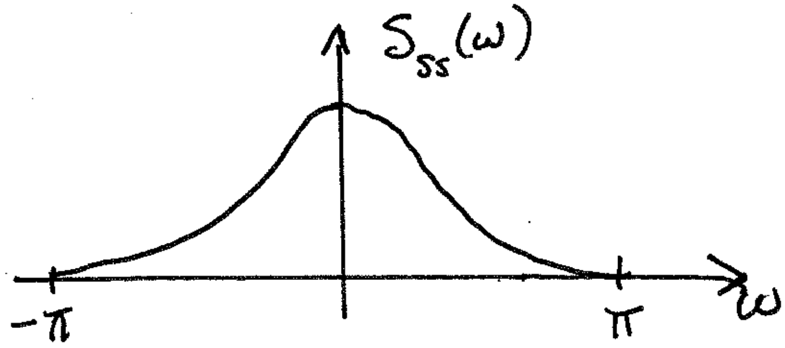
$m_s = 0$ $\qquad R_{ss}[m],\ S_{ss}(\omega)$ arbitrary

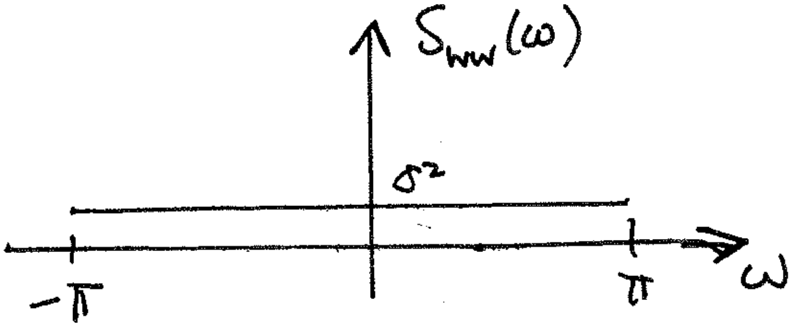
noise:

$m_w = 0$ $\qquad R_{ww}[m] = \sigma^2 \delta[m]$

$S_{ww}(\omega) = \sigma^2$ const.

Consider the case





Define the noise removal filter

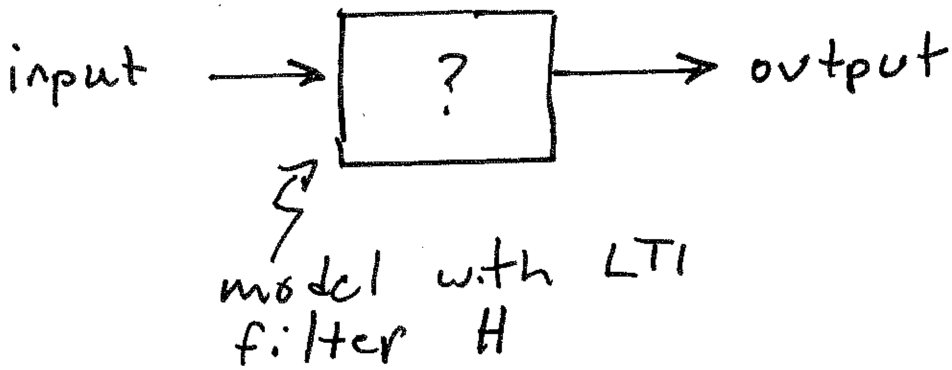
$$X[n] \longrightarrow \boxed{H} \longrightarrow Y[n]$$

$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

How to choose ω_c?

Ex. System Identification

Suppose that we wish to model an <u>unknown</u> physical system with an LTI filter.



How can we "identify" the filter H?

White Noise Probing:

$$\text{input} \equiv x[n] = \text{GWN}$$

$$\Rightarrow \quad x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$m_y = 0$$

$$R_{yy}[m] = \sum_{k=-\infty}^{\infty} \sigma^2 h[k]\, h[m+k]$$

$$S_{yy}(\omega) = \sigma^2 \left| H(\omega) \right|^2$$

close, but only tells us magnitude

Stationary Processes

Properties of Correlation Function:

(1) $|R_{xx}[m]| \leq R_{xx}[0]$

(2) $R_{xx}[-m] = R_{xx}[m]$

(3) power of process $x[n]$ is given by $R_{xx}[0]$

proof of (1): $(x[n] - x[n+m])^2 \geq 0$

$$\Rightarrow E[(x[n] - x[n+m])^2] \geq 0$$

similarly, $(x[n] + x[n+m])^2 \geq 0$

$$\Rightarrow E[(x[n] + x[n+m])^2] \geq 0$$

proof of (2):

$$R_{xx}[m] = E[x[n]\,x[n+m]]$$

for any n!

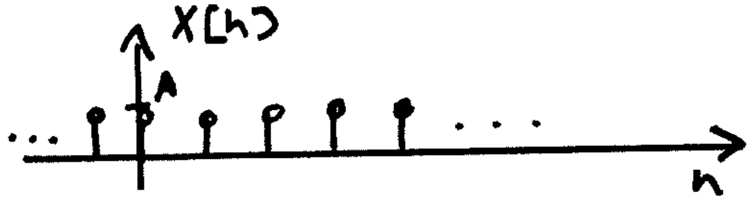
$$R_{xx}[-m] = E[x[n]\,x[n-m]]$$

take $n = n' - m$

$$R_{xx}[m] = E[x[n'-m]\,x[n']] = R_{xx}[-m]$$

Correlation

Total Correlation:



$$X[n] = A \,, \qquad P_A(a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-a^2/2\sigma^2}$$

mean:

correlation function:

Zero Correlation:

Strong Correlation:

$$w[n] \longrightarrow \boxed{\frac{1}{1-ae^{-j\omega}}} \longrightarrow x[n]$$

GWN

$$x[n] = a\,x[n-1] + w[n] \quad ; \quad 0 < a < 1$$

impulse response $h[n] =$

mean:

Correlation function:

Stationarity

Let $x[n]$ be a random process.

Then we can talk about

$$Pr\left(|x[n]| < a \right)$$

or $$Pr\left(a < x[n] < b \right).$$

We can also study the joint probability distributions for two or more samples:

$$Pr\left(a < x[n] < b,\ c < x[n+m] < d \right)$$

or $$Pr\left(|x[n]| \le a,\ |x[n+1]| \le b,\ |x[n+2]| \le c \right)$$

Strict-Sense Stationarity (SSS)

Strict-sense stationarity means that these joint probability distributions are time-invariant.

ex. SSS

$$\Pr\left(a < x[n] < b,\ c < x[n+m] < d\right)$$
$$= \Pr\left(a < x[n+k] < b,\ c < x[n+k+m] < d\right)$$

for every $k \in \mathbb{Z}$.

SSS & LTI Systems

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

SSS

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$\Pr\left(a \le y[n] \le b\right) = \Pr\left(a \le \sum_k h[k]\, x[n-k] \le b\right)$$
$$= \Pr\left(a \le \sum_k h[k]\, x[n+m-k] \le b\right) \quad \text{by SSS of } x[n]$$
$$= \Pr\left(a \le y[n+m] \le b\right)$$

Wide-Sense Stationary (WSS)

(1) $E\left[x[n]\right] = m_x$ const.

(2) $E\left[x[n]x[n+m]\right] = R_{xx}[m]$

(1) $\Rightarrow$ average value of process is time-invariant

(2) $\Rightarrow$ average power of process and correlation between samples is time-invariant

WSS is <u>weaker</u> than SSS:

$$\text{SSS} \underset{\not\Leftarrow}{\Rightarrow} \text{WSS}$$

WSS tells us that <u>average</u> behavior of single samples and products of samples is time-invariant.

SSS tells us that the joint probability distributions for arbitrary collections of samples are time-invariant.

WSS & LTI Systems

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$x[n]$: WSS; H: LTI $\Rightarrow$ WSS output

In most cases and applications WSS sufficient for analysis purposes.

(1) $E[x[n]] = m_x \Rightarrow E[y[n]] = m_y$ const.

(2) $$\underbrace{E[x[n]\,x[n+m]] = R_{xx}[m]}_{\text{correlation structure}}$$

correlation structure is sufficient to describe frequency content of input

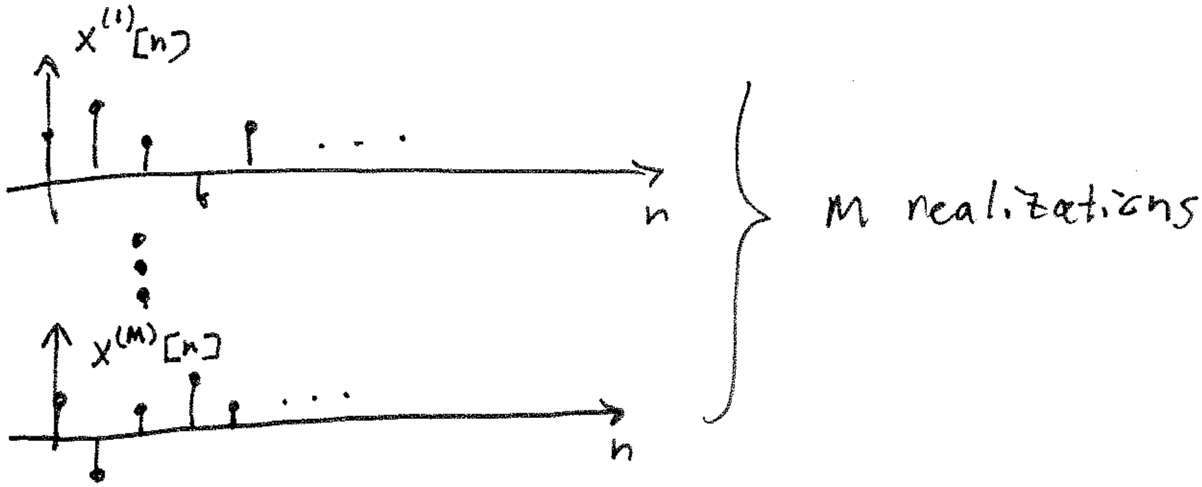
$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}[m]\, e^{-j\omega m}$$

Thus, WSS is sufficient for frequency-domain analysis of random processes through LTI systems.

Estimating Means and Autocorrelations from Data

Suppose that we make several independent observations of a random process:

$$x^{(i)}[n], \qquad \underbrace{i = 1, \ldots, M}_{\text{realizations}}$$



We can estimate the mean function using

$$\boxed{\hat{m}_x[n] = \frac{1}{M} \sum_{i=1}^{M} x^{(i)}[n]}$$

$$\hat{m}_x[n] \longrightarrow m_x[n] \quad \text{as } M \to \infty$$

We can estimate the autocorrelation function according to

$$\boxed{\hat{R}_{xx}[n, n+m] = \frac{1}{M} \sum_{i=1}^{M} x^{(i)}[n]\, x^{(i)}[n+m]}$$

If the process is <u>stationary</u>, then we can estimate the mean and auto correlation from a <u>single</u> realization using <u>time-averaging</u>.

$$\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\hat{R}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n]\, x[n+m]$$

$$\left.\begin{array}{l} \hat{m}_x \longrightarrow m_x \\ \hat{R}_{xx}[m] \longrightarrow R_{xx}[m] \end{array}\right. \quad \text{as } N \longrightarrow \infty$$