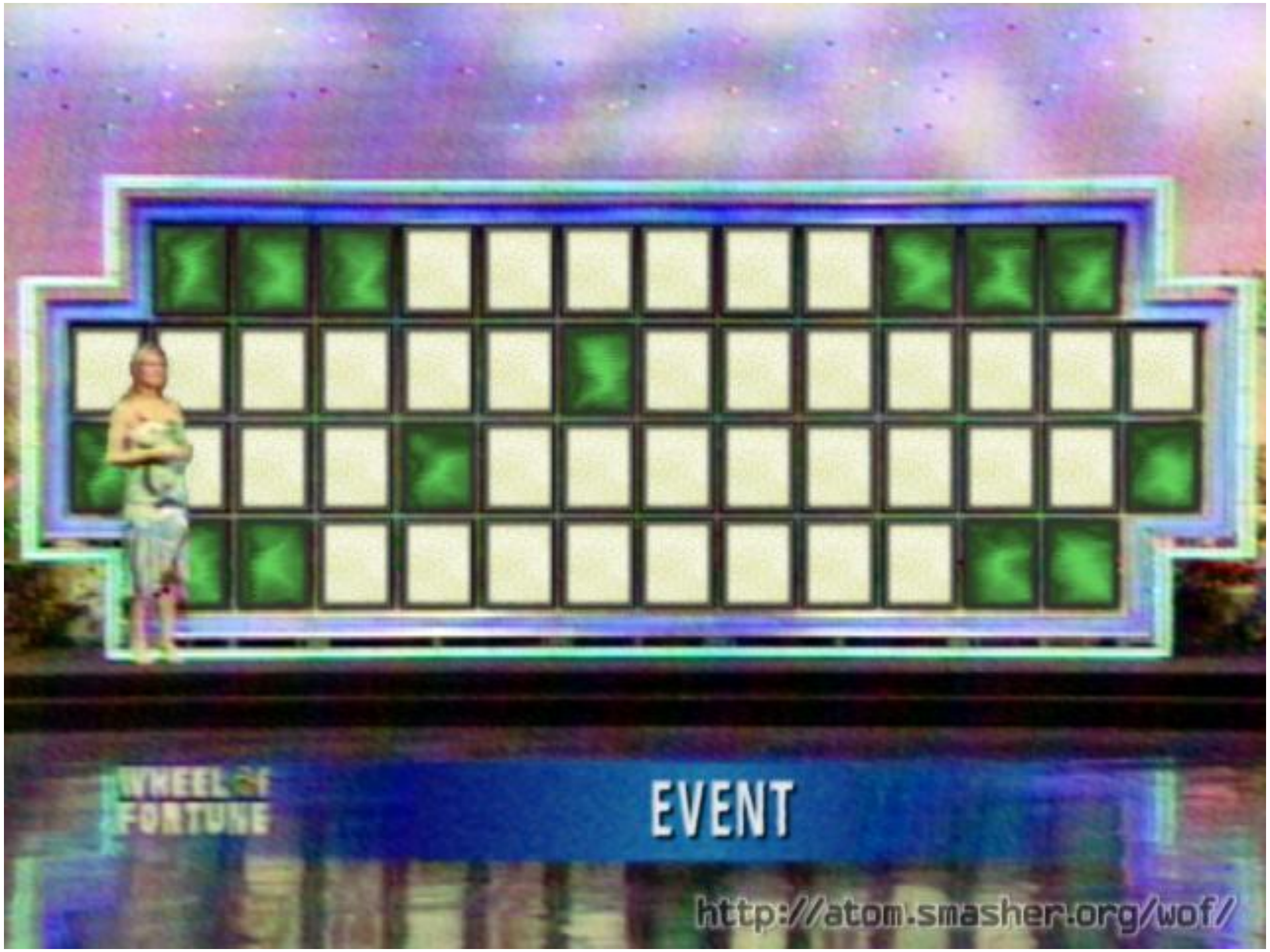
# *Active Sensing*

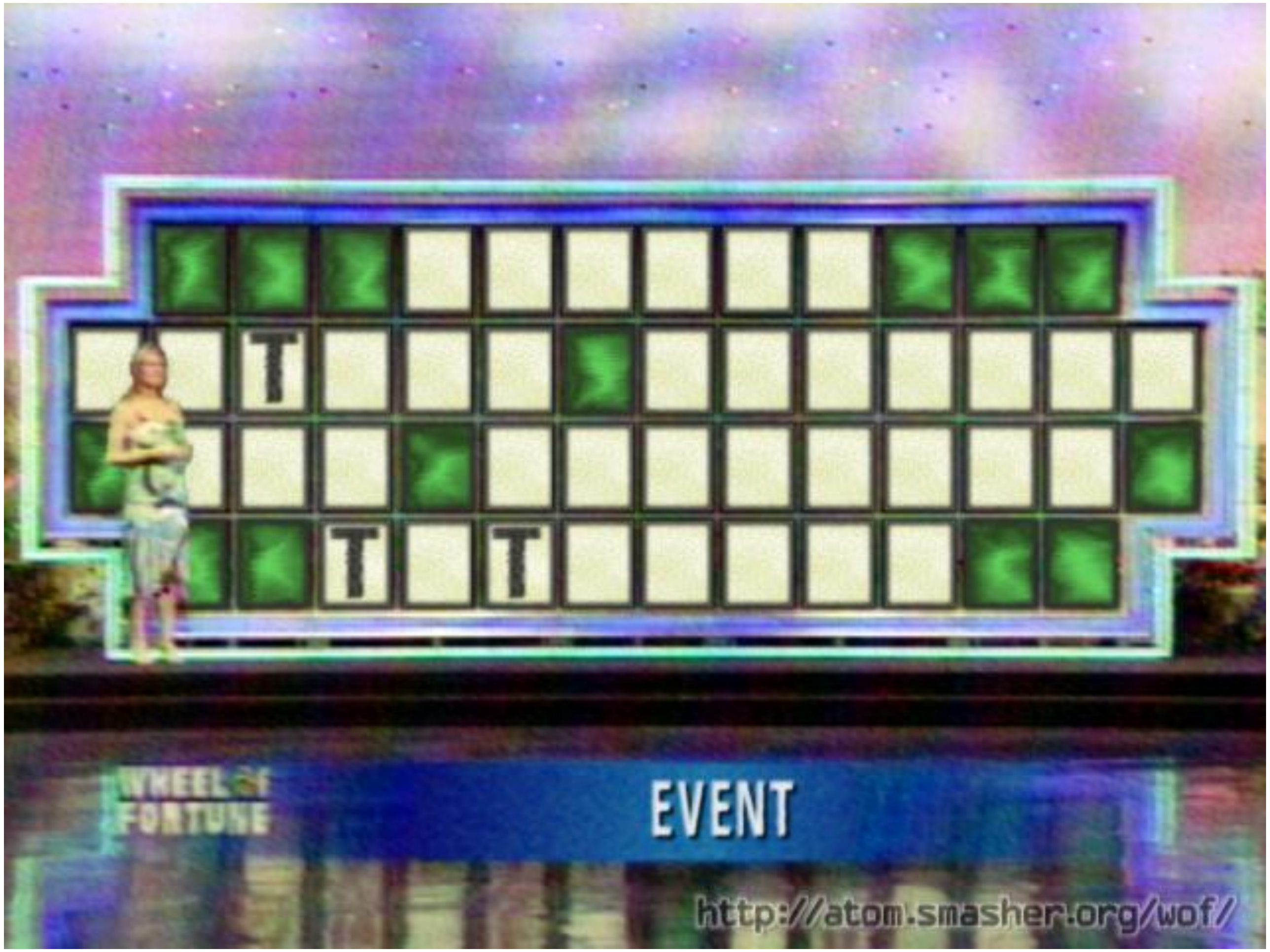
*What is Active Sensing?*

*For some, it’s a hobby!*

# “Object” to be Inferred



# "Measurement"



# “Measurement”

# “Feedback” Guides New “Measurements”

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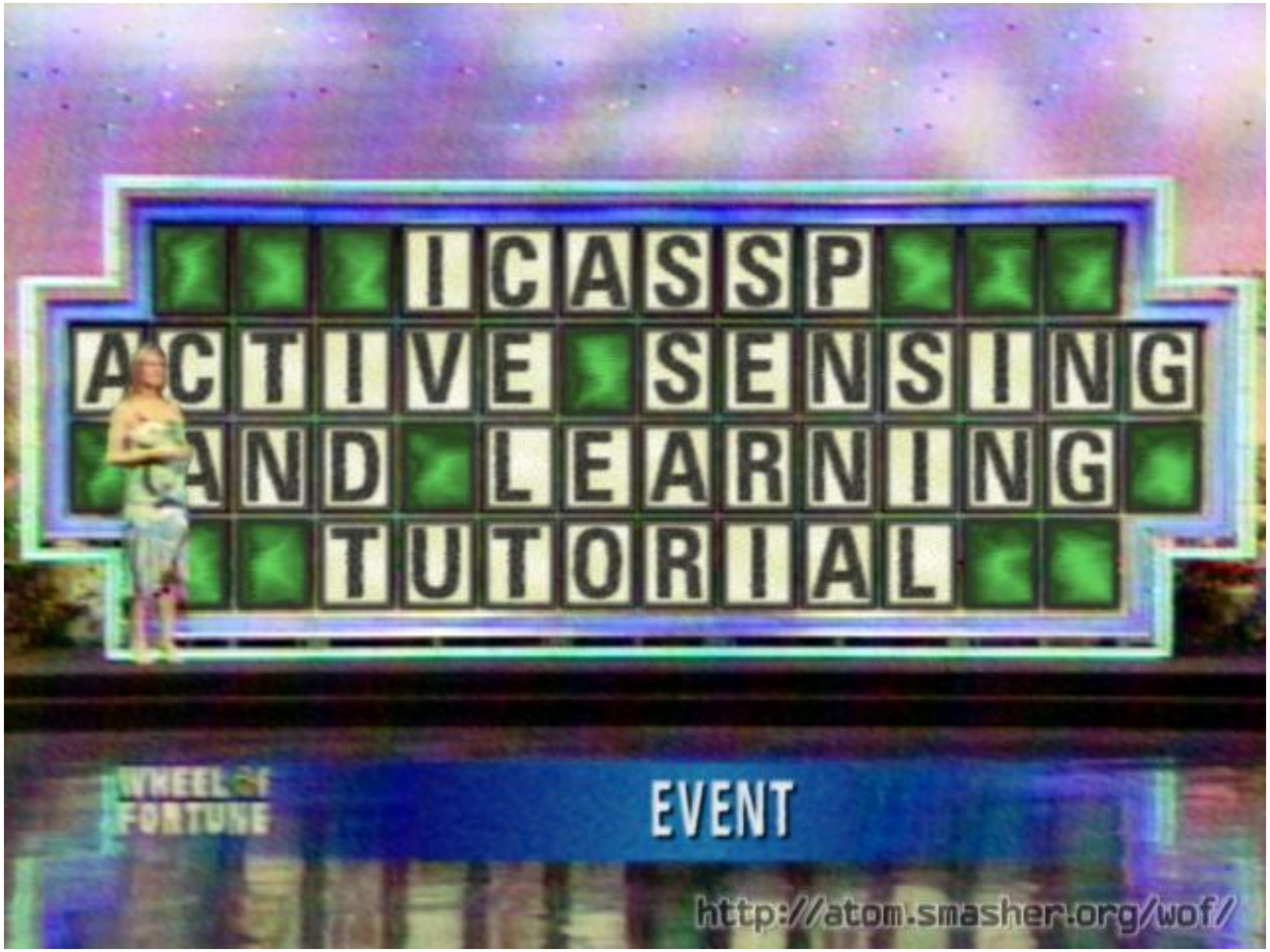
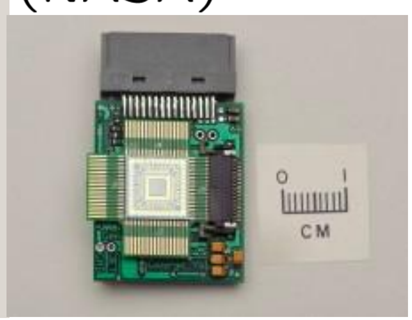


# *Background and Motivation*

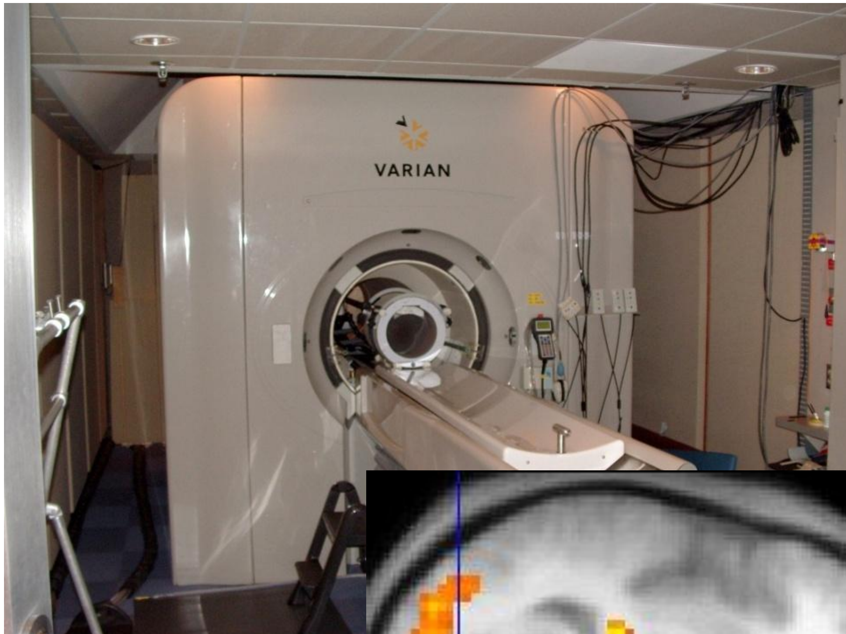
# Sensors, Sensors Everywhere

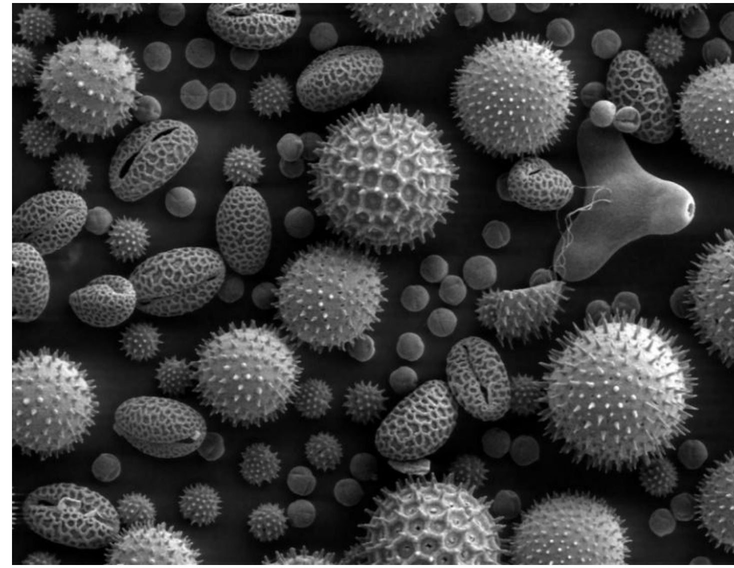

Images, sound, GPS, accelerometer, proximity,...
(Apple)

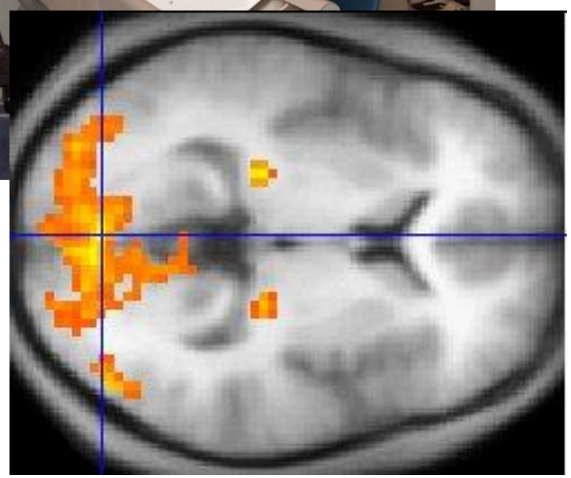
$NH_3$, $Cl_2$,...
(NASA)



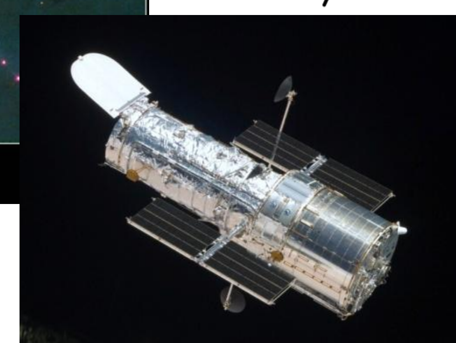

# Integral to Science, Engineering, Discovery



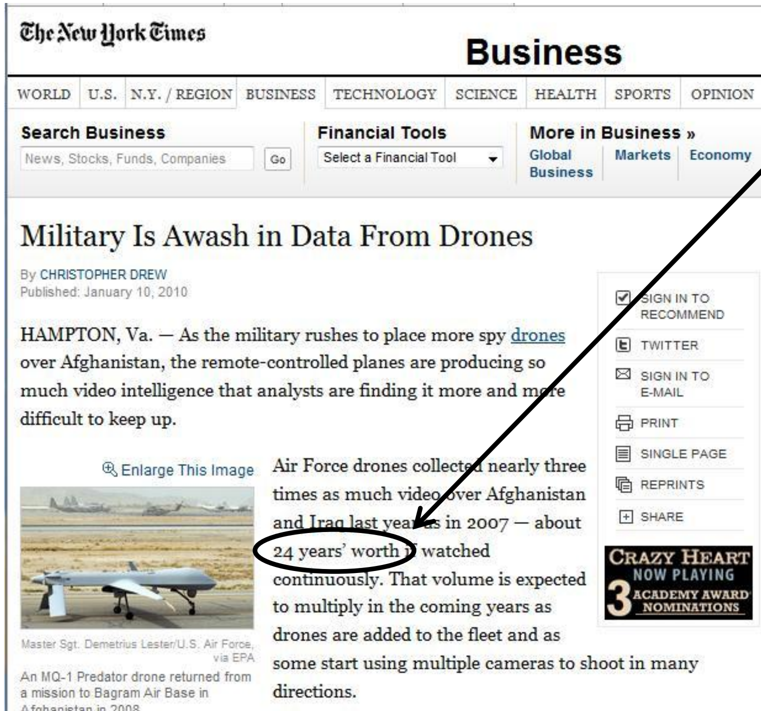






NASA/ESA



# Inevitable Data Deluge!



The New York Times

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## Military Is Awash in Data From Drones

By CHRISTOPHER DREW
Published: January 10, 2010

HAMPTON, Va. — As the military rushes to place more spy drones over Afghanistan, the remote-controlled planes are producing so much video intelligence that analysts are finding it more and more difficult to keep up.

Enlarge This Image

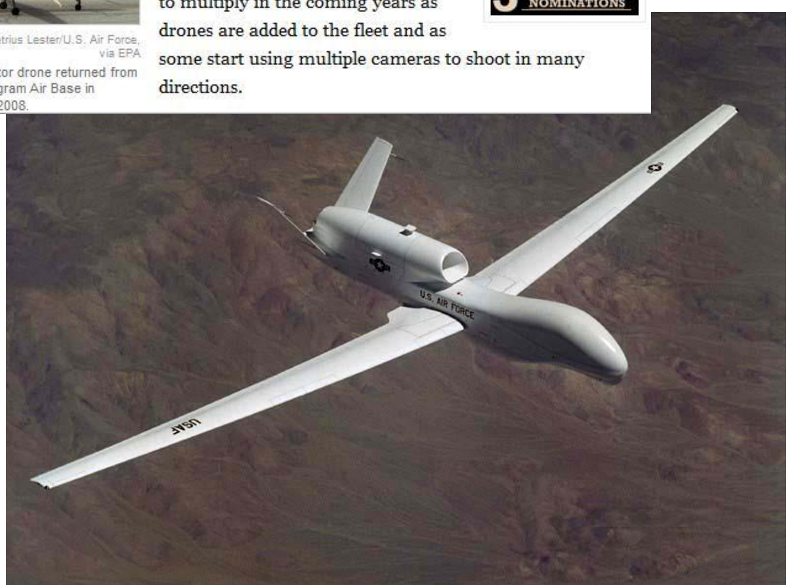
Master Sgt. Demetrius Lester/U.S. Air Force, via EPA
An MQ-1 Predator drone returned from a mission to Bagram Air Base in Afghanistan in 2008.

Air Force drones collected nearly three times as much video over Afghanistan and Iraq last year as in 2007 — about 24 years' worth if watched continuously. That volume is expected to multiply in the coming years as drones are added to the fleet and as some start using multiple cameras to shoot in many directions.

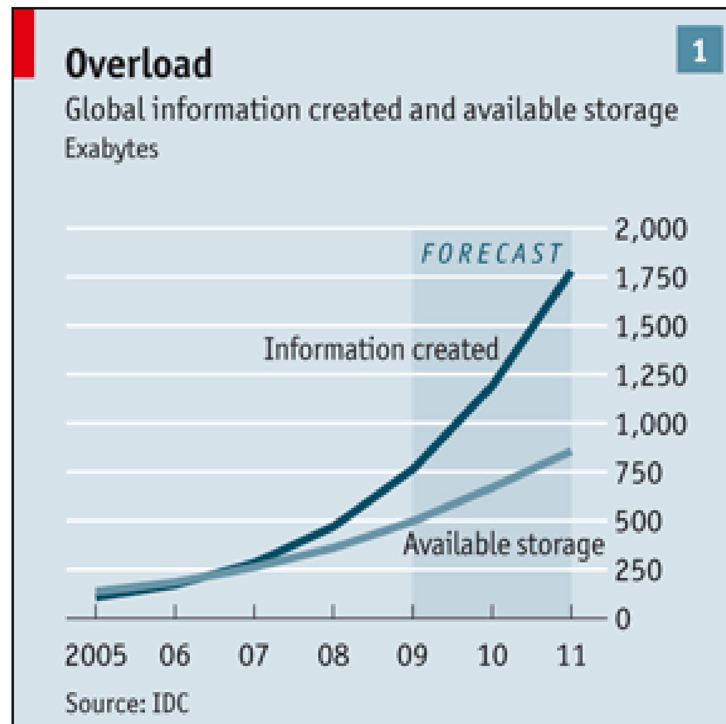
SIGN IN TO RECOMMEND
TWITTER
SIGN IN TO E-MAIL
PRINT
SINGLE PAGE
REPRINTS
SHARE



24 years/year !



YouTube Upload Rate (March 2010):
24 hours/minute ?!?!
(www.webpronews.com)




The Economist, February 2010

# Challenges for Sensing/Processing Systems

## Technology:

technologically impossible to sense/observe everything, everywhere, all the time
$\Rightarrow$incomplete, missing, or indirect data are the norm

## Uncertainty:

experiments/measurements are noisy, corrupted or unreliable!
$\Rightarrow$info-processing and decision-making must be robust to uncertainty

## Complexity:

systems can be ultra high-dimensional
$\Rightarrow$modeling/approximation is formidable, mathematically & computationally

## Diversity:

data from disparate sources
$\Rightarrow$integration of info from sensors, experiments, databases, human intel, etc.

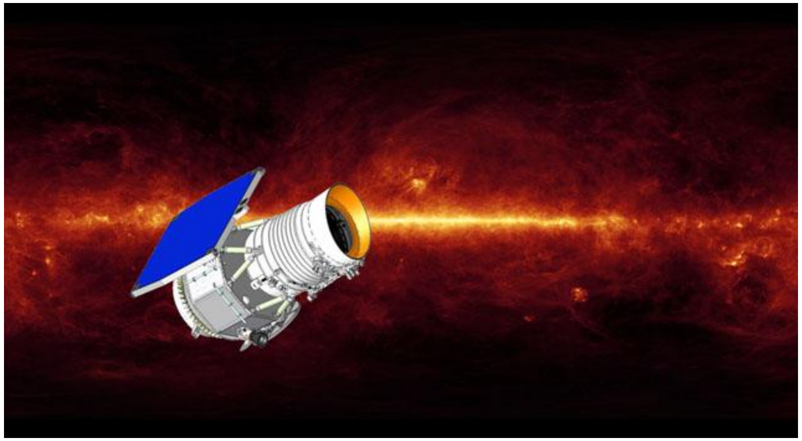
## Approaches:

$\Rightarrow$Rethink Traditional Sensing Strategies
$\Rightarrow$Integration of Sensing and Processing

*Active Sensing*

*-- An Engineering Example --*

# Wide-field Infrared Survey Explorer (WISE)



→ Need to shield IR (heat) from its own instruments

→ Sensitive instruments housed in solid hydrogen

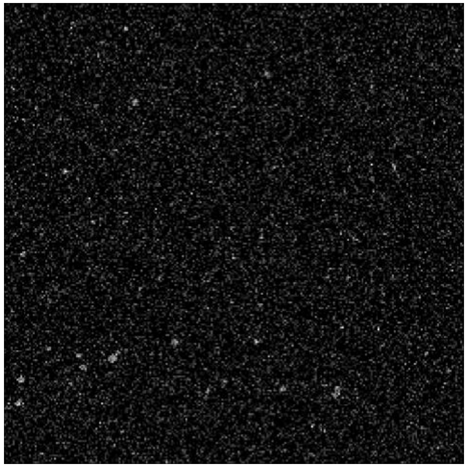
→ Expected lifetime: 10 months!

WISE Mission:
(http://www.nasa.gov/mission_pages/WISE/mission/index.html)

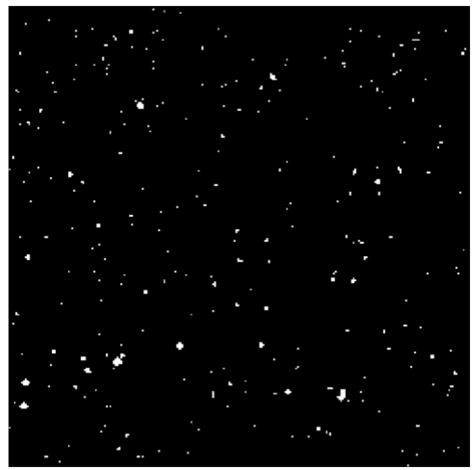
"…the infrared surveyor will spend six months mapping the whole sky. It will then begin a second scan to uncover even more objects and to look for any changes in the sky that might have occurred since the first survey. This second partial sky survey will end about three months later when the spacecraft's frozen-hydrogen cryogen runs out…"

Fornax Galaxy Cluster, Feb. 17 2010

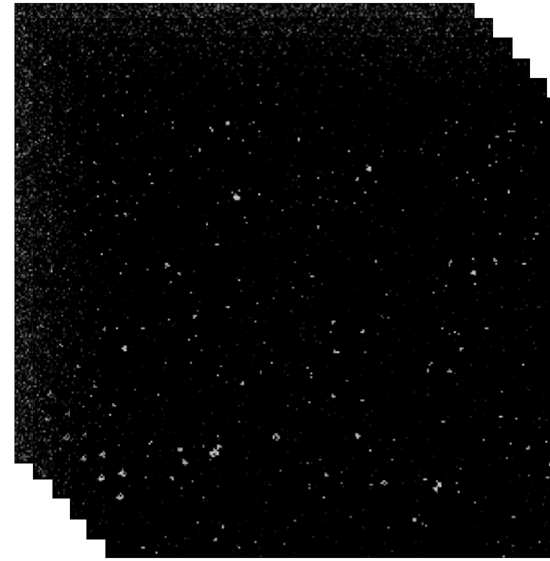
# Astronomical Imaging "On a Budget"



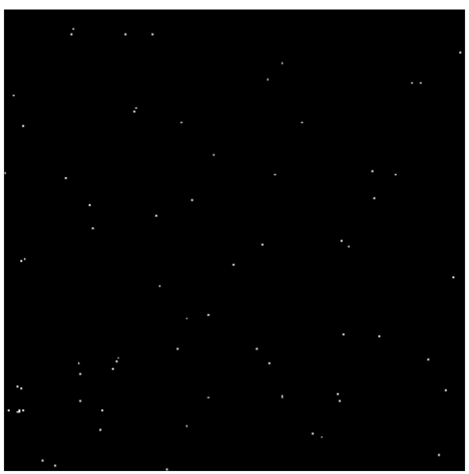
Noisy, non-adaptive sampling



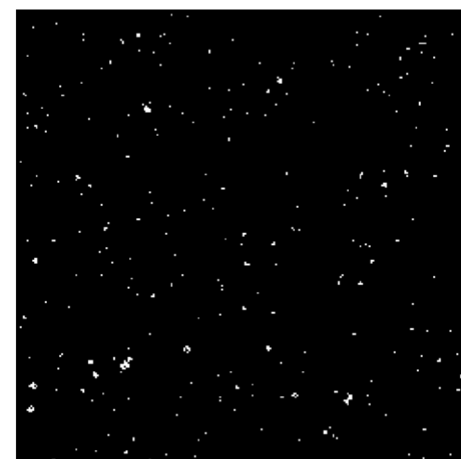
original signal
(~0.8% non-zero components)



Noisy adaptive sampling



Recovery from non-adaptive samples
(1/20 "discoveries" are errors)

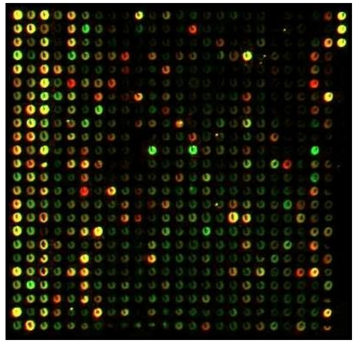


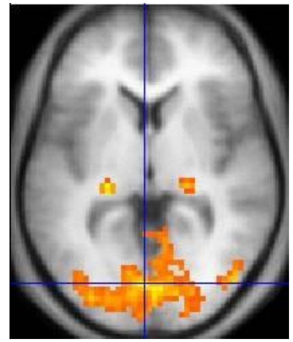
Recovery from adaptive samples
(1/20 "discoveries" are errors)

# *Active Sensing for Sparse Recovery*

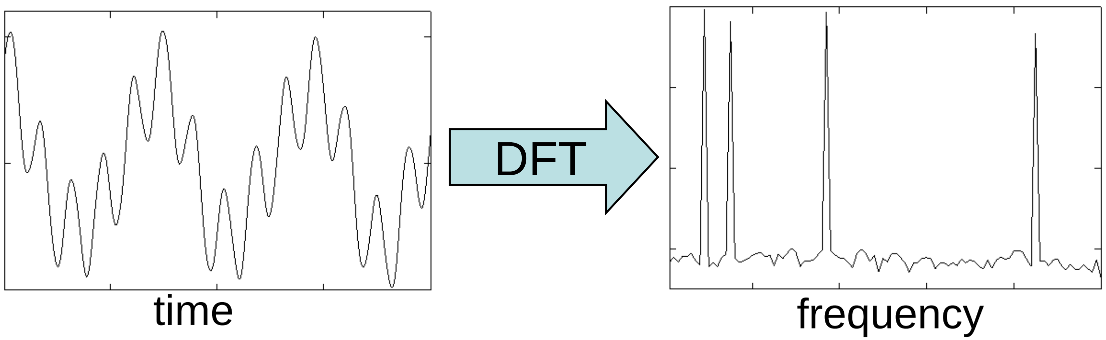
## *-- Preliminaries and Formalization --*

# Sparsity

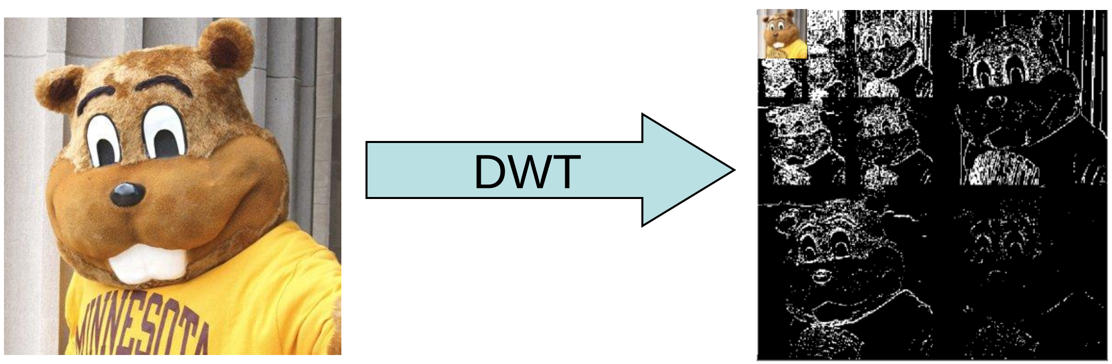




Many signals exhibit sparsity in the canonical or "pixel" basis




Communication signals often have sparse frequency content



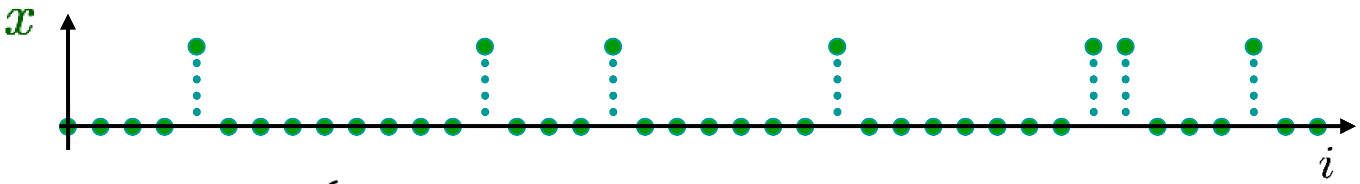

Natural images often have sparse wavelet representations

Notion of sparsity extends to other models (task-specific dictionaries, manifolds, etc.)

Sparse Recovery Goal: Identify the locations of the nonzeros!

# A Sparse Signal Model

Signals of interest are vectors $x \in \mathbb{R}^n$

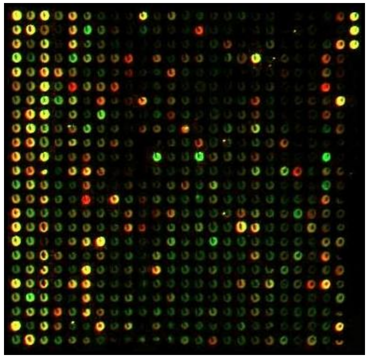
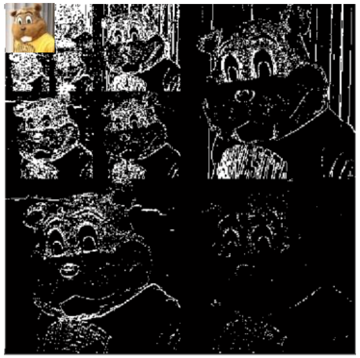
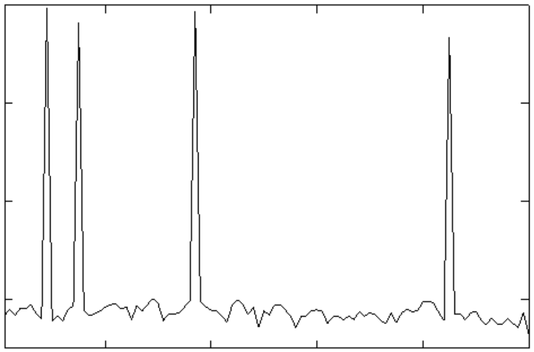



$$x_i = \begin{cases} \mu > 0 & i \in \mathcal{S} \\ 0 & i \notin \mathcal{S} \end{cases}$$

signal support set

$$\text{Sparse} \Leftrightarrow |\mathcal{S}| \ll n$$

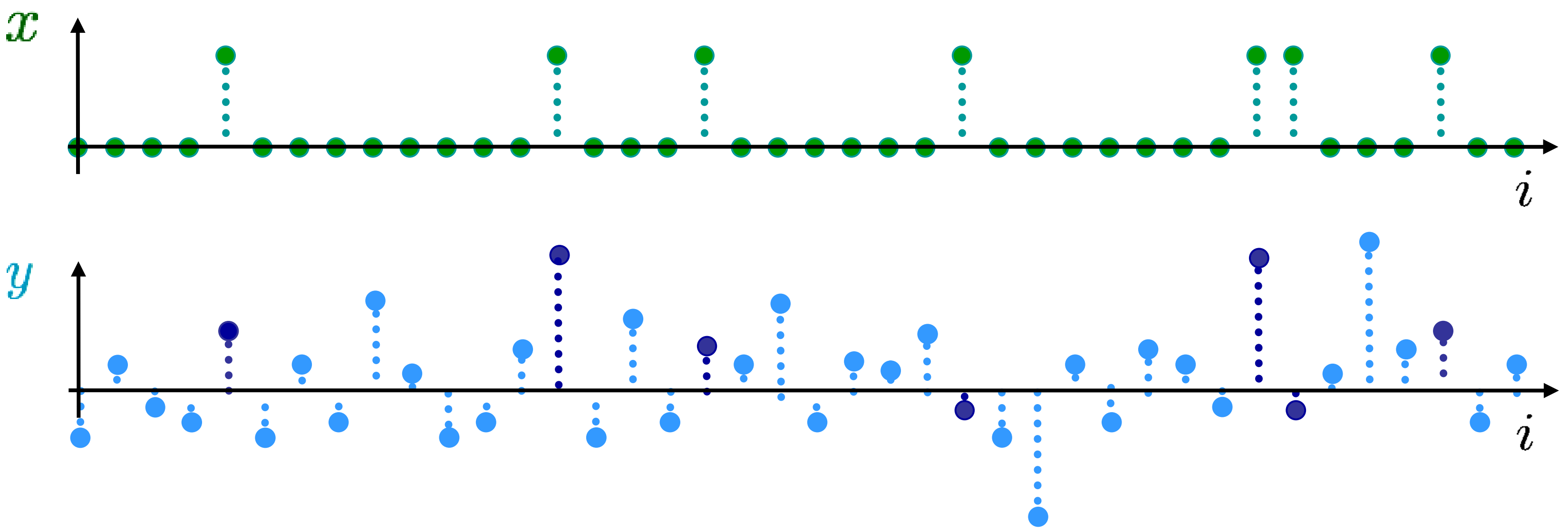
number of nonzero
signal components

# Non-adaptive Sampling

Non-adaptive observations:

$$y_i = x_i + z_i$$
$$z_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$$



# Alternative: Sequential Experimental Design

$$y_{i,j} = \begin{cases} x_i + \gamma_{i,j}^{-1/2} \, z_{i,j}, & \gamma_{i,j} > 0, \; i = 1, \dots, n, \quad j = 1, \dots, k \\ 0 & \gamma_{i,j} = 0, \; i = 1, \dots, n, \quad j = 1, \dots, k \end{cases}$$

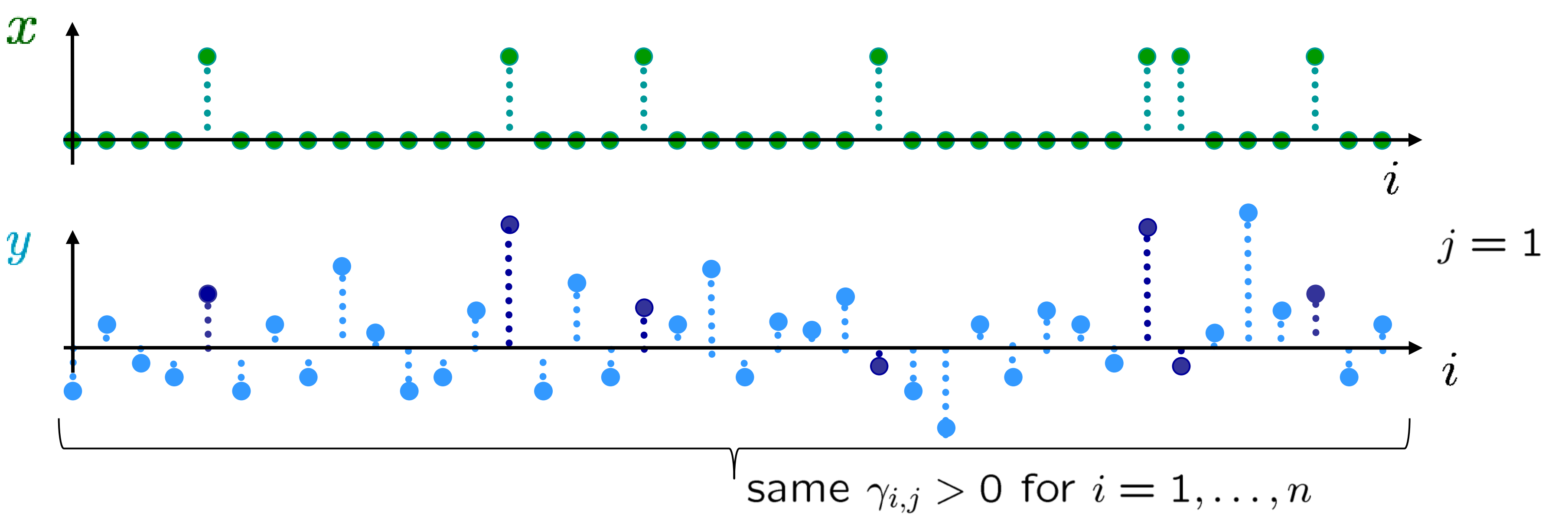
$z_{i,j} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$

$j$ indexes the observation step

$k$ is the total number of steps

$\gamma_{i,j} \geq 0$ is the *precision* of observation $y_{i,j}$

$$\Rightarrow y_{i,j} \sim \mathcal{N}\left(x_i, 1/\gamma_{i,j}\right) \text{ when } \gamma_{i,j} \neq 0$$



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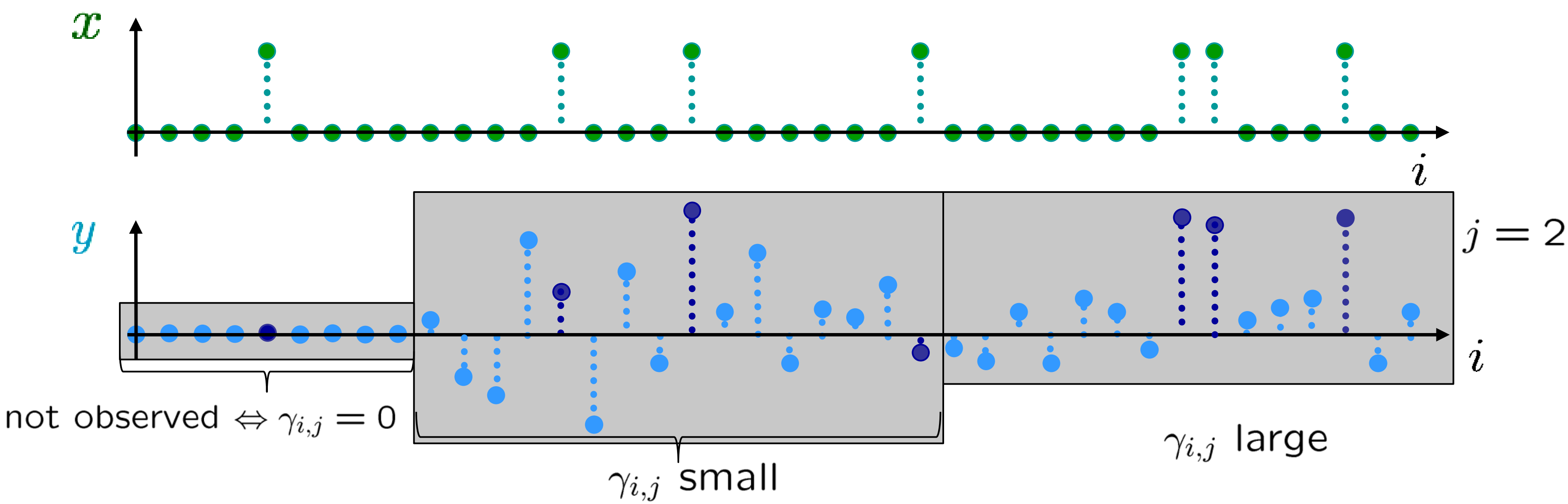

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$$\Rightarrow y_{i,j} \sim \mathcal{N}\left(x_i, 1/\gamma_{i,j}\right) \;\; \text{when } \gamma_{i,j} \neq 0$$

Precision is increased (decreased) by:
– Averaging more (fewer) repeated samples
– Longer (shorter) observation times

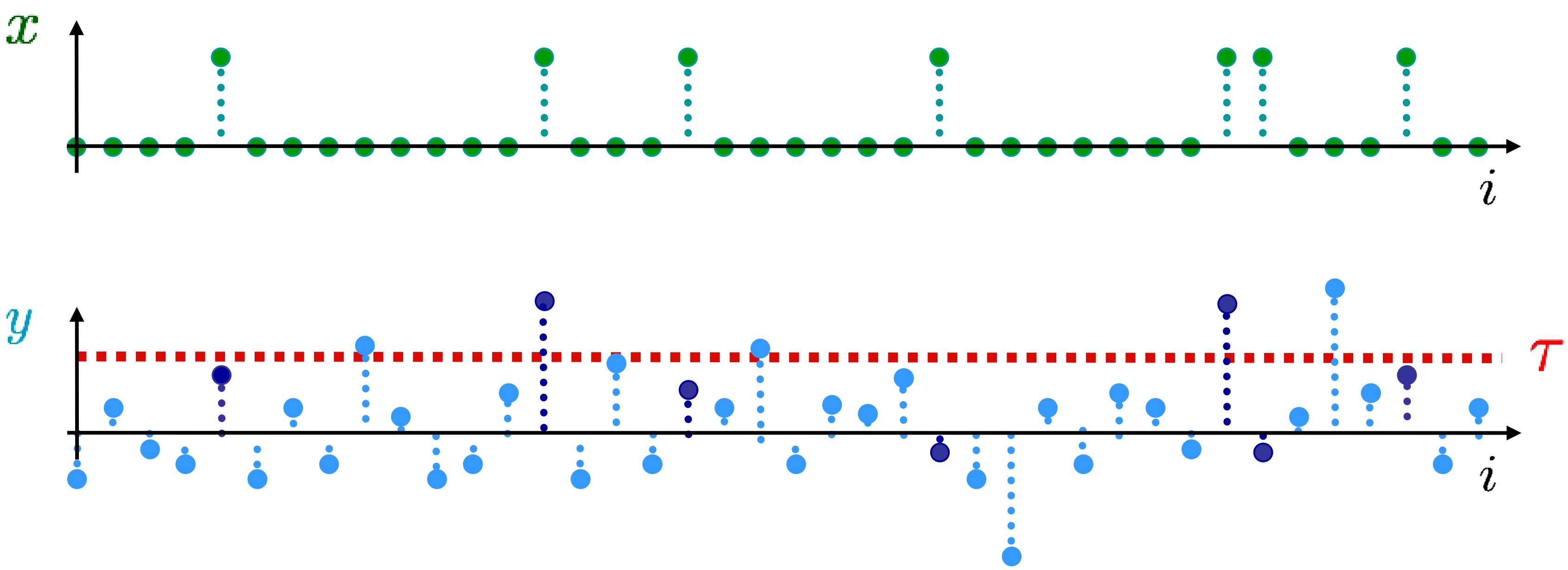
*Total precision* subject to a global constraint:

$$\sum_{j=1}^{k} \sum_{i=1}^{n} \gamma_{i,j} \leq R(n)$$

Proportional to total # samples, total time, total energy, cryogen life, etc.

# Support Recovery

Goal: Estimate the signal support set $\mathcal{S} := \{i \in \{1, \ldots, n\} : x_i \neq 0\}$



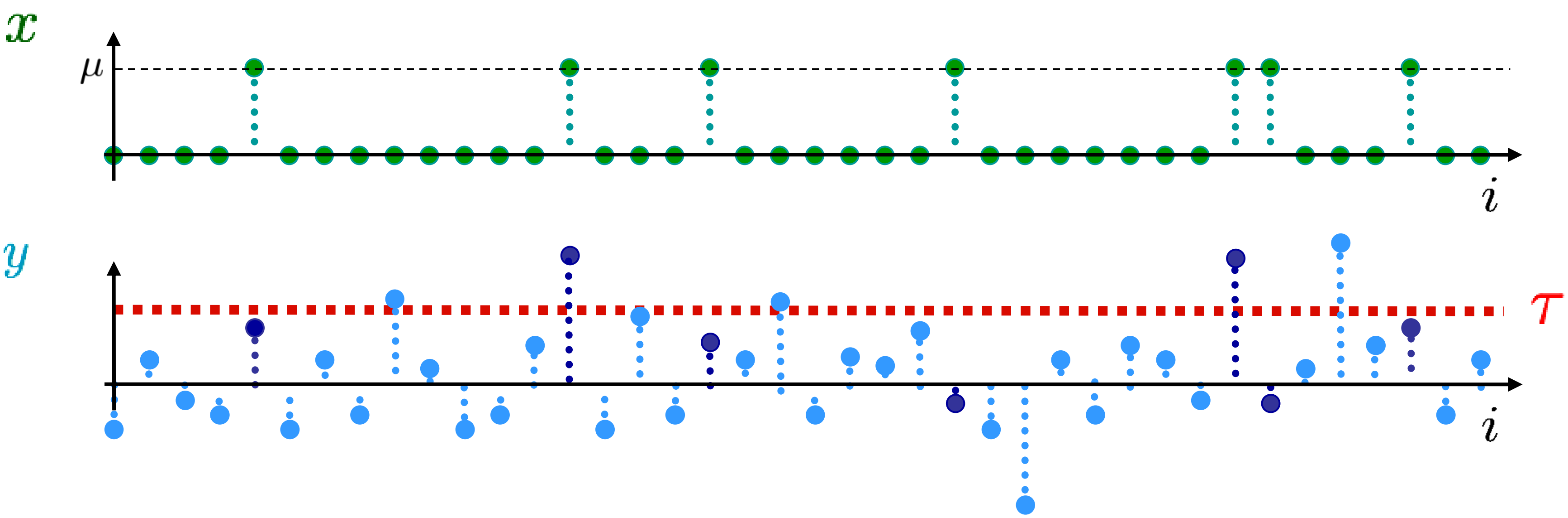

**Definition**: A *threshold test* is an estimator of the form

$$\widehat{\mathcal{S}}_\tau(y) := \{i \in \{1, \ldots, n\} : y_i > \tau\}$$

# *A Simple Active Sensing Approach*

# Recovery From Non-Adaptive Measurements

Goal: Estimate $\mathcal{S}$ with *no* errors (ie, $\widehat{\mathcal{S}} = \mathcal{S}$) from noisy measurements
How large must amplitude $\mu$ be?

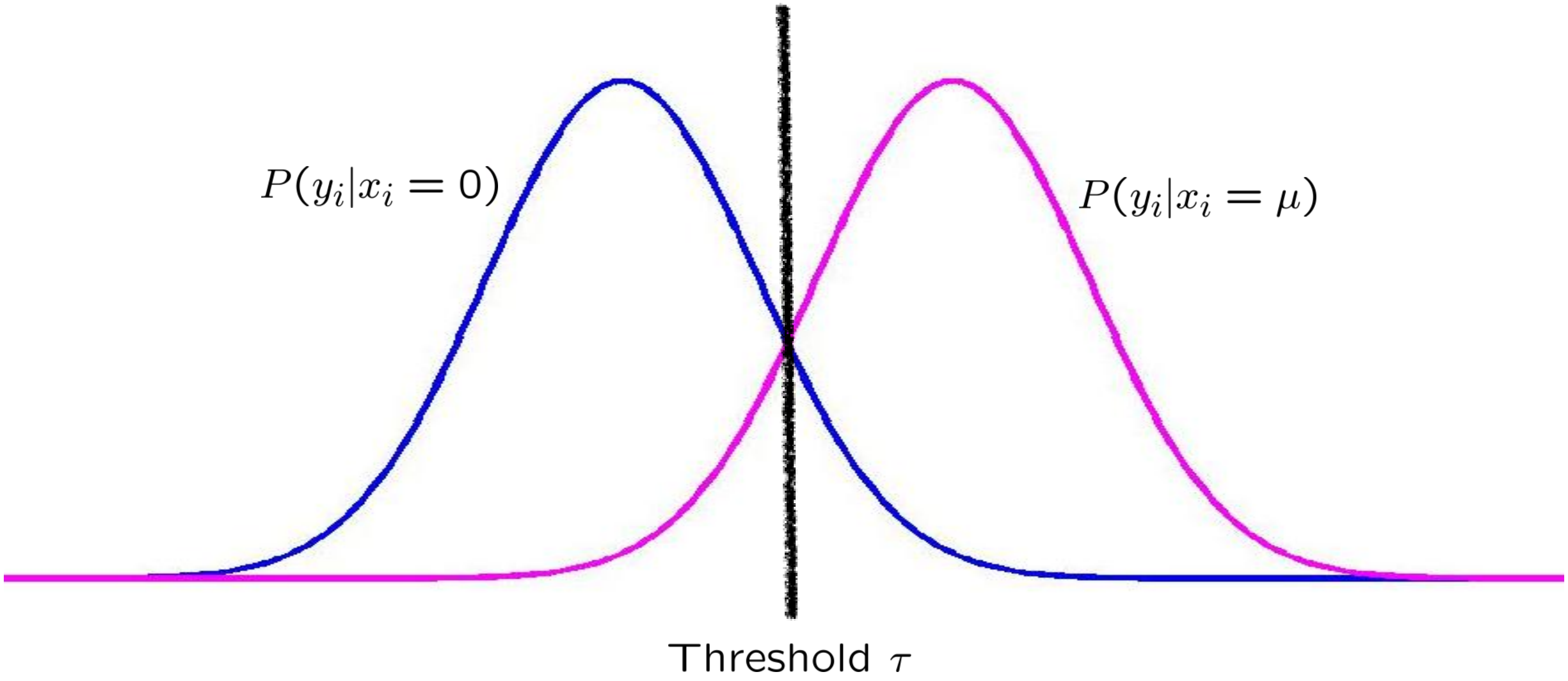



Exact support recovery $\Leftrightarrow \begin{cases} y_i > \tau \text{ for all } i \in \mathcal{S} \\ y_i < \tau \text{ for all } i \in \mathcal{S}^c \end{cases}$

Let $|\mathcal{S}| = s$, then $n - s$ components of $x$ are equal to zero.

# Fundamentally a Multiple Hypothesis Test

Test signal present vs. signal absent at each coordinate:



# Non-Adaptive Support Recovery

How large must $\mu$ be to ensure probability of error tends to zero?

$$\mathbb{P}(\widehat{\mathcal{S}} \neq \mathcal{S}) \leq (n-s)\mathbb{P}(y_i > 0 | x_i = 0) + s\ \mathbb{P}(y_i < 0 | x_i = \mu)$$

$$\leq \underbrace{\frac{n-s}{2} \exp\left(-\frac{\tau^2}{2}\right)} + \underbrace{\frac{s}{2} \exp\left(-\frac{(\mu-\tau)^2}{2}\right)}$$

Want each term to tend to zero

$$\tau \gtrsim \sqrt{2 \log(n-s)} \qquad \mu \gtrsim \tau + \sqrt{2 \log s}$$

$$\mu \gtrsim \sqrt{2 \log(n-s)} + \sqrt{2 \log s}$$

Necessary condition: $\mu \gtrsim \sqrt{\log n}$

# Adaptive Approach: Sequential Thresholding

**Sequential Thresholding**

initialize: $\mathcal{S}_0 = \{1, \ldots, n\}$, $\gamma_{i,j}^{-1} = 2$

for $j = 1, \ldots, k$

1) measure: $y_{i,j} \sim \mathcal{N}(x_i, 2)$ , $i \in \mathcal{S}_{j-1}$

2) threshold: $\mathcal{S}_j = \{i : y_{i,j} \geq 0\}$

end

output: $\mathcal{S}_k = \{i : y_{i,k} > 0\}$

total precision budget: $\mathbb{E}\left[\sum_{i,j} \gamma_{i,j}\right]$

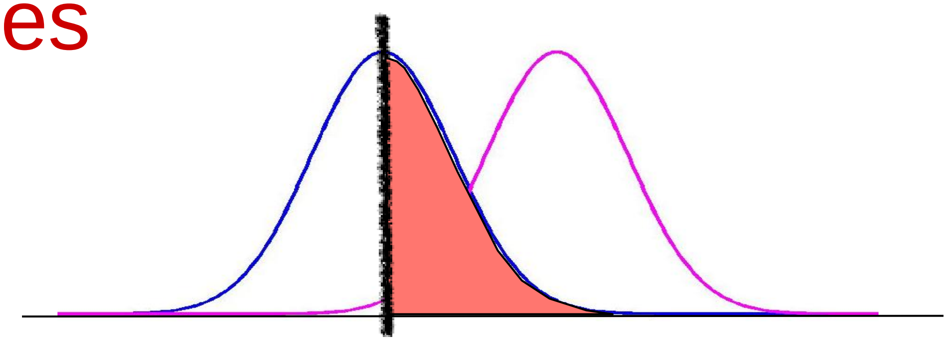
$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{k} \mathbb{E}|\mathcal{S}_{j-1}| \\
&\leq \frac{1}{2}\sum_{j=1}^{k}\left(\frac{n-s}{2^{j-1}} + s\right) \\
&\leq n - s + ks \approx n
\end{aligned}
$$

(when $n \gg s$)

probability of error:

$$
\begin{aligned}
\mathbb{P}(\mathcal{S}_k \neq \mathcal{S}) &= \mathbb{P}\left(\{\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset\} \cup \{\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset\}\right) \\
&\leq \mathbb{P}\left(\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset\right) + \mathbb{P}\left(\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset\right)
\end{aligned}
$$

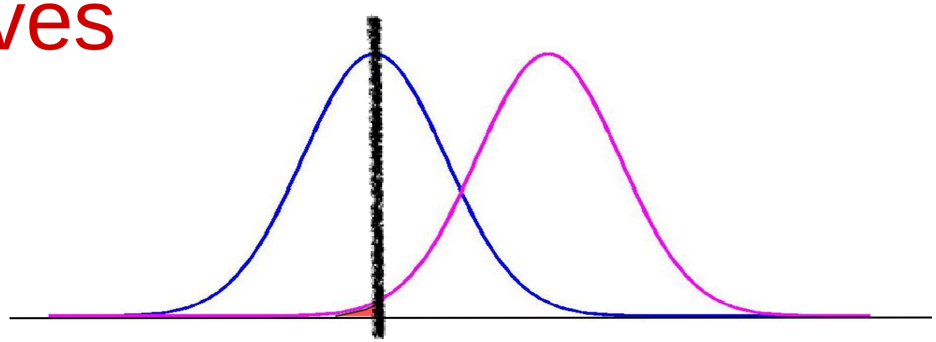
# False Positives



$$\mathbb{P}(\mathcal{S}_k \neq \mathcal{S}) \leq \mathbb{P}\left(\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset\right) + \mathbb{P}\left(\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset\right)$$

$$\begin{aligned}
\mathbb{P}\left(\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset\right) &= \mathbb{P}\left(\bigcup_{i \notin \mathcal{S}} \bigcap_{j=1}^{k} y_{i,j} > 0\right) \\
&\leq \sum_{i \notin \mathcal{S}} \mathbb{P}\left(\bigcap_{j=1}^{k} y_{i,j} > 0\right) \\
&= \sum_{i \notin \mathcal{S}} 2^{-k} = \frac{n-s}{2^k}
\end{aligned}$$

# False Negatives



$$\mathbb{P}(\mathcal{S}_k \neq \mathcal{S}) \leq \mathbb{P}\left(\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset\right) + \mathbb{P}\left(\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset\right)$$

$$\begin{aligned}\mathbb{P}\left(\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset\right) &= \mathbb{P}\left(\bigcup_{j=1}^{k} \bigcup_{i \in \mathcal{S}} y_{i,j} < 0\right) \\ &\leq \frac{ks}{2} \exp\left(-\frac{\mu^2}{4}\right)\end{aligned}$$

# Probability of Error Bound

$$\mathbb{P}(\mathcal{S}_k \neq \mathcal{S}) \leq \mathbb{P}(\mathcal{S}^c \cap \mathcal{S}_k \neq \emptyset) + \mathbb{P}(\mathcal{S} \cap \mathcal{S}_k^c \neq \emptyset)$$

$$\leq \frac{n-s}{2^k} + \frac{ks}{2} \exp\left(-\frac{\mu^2}{4}\right)$$

$$= \frac{n-s}{2^k} + \frac{1}{2} \exp\left(-\frac{(\mu^2 - 4\log(ks))}{4}\right)$$

Choose $k = \log_2 n^{1+\epsilon}$ and consider high-dimensional limit...

$$\mathbb{P}(\mathcal{S}_k \neq \mathcal{S}) \leq \underbrace{\frac{n-s}{2^k}}_{\to 0} + \frac{1}{2} \exp\left(-\frac{(\mu^2 - 4\log(s(1+\epsilon)\log_2 n))}{4}\right)$$

Probability of error goes to zero if

$$\mu \gtrsim \sqrt{4\log(s(1+\epsilon)\log_2 n)}$$

(Malloy & Nowak, 2011)

# Improvements Through Sequential Design

**non-sequential:** $\mu \gtrsim \sqrt{2\log(n-s)} + \sqrt{2\log s}$ (necessary)

**sequential thresholding:** (sufficient)

$$\begin{aligned} \mu &\gtrsim \sqrt{4\log(s(1+\epsilon)\log_2 n)} \\ &\cong 2\sqrt{\log s + \log\log_2 n} \end{aligned}$$

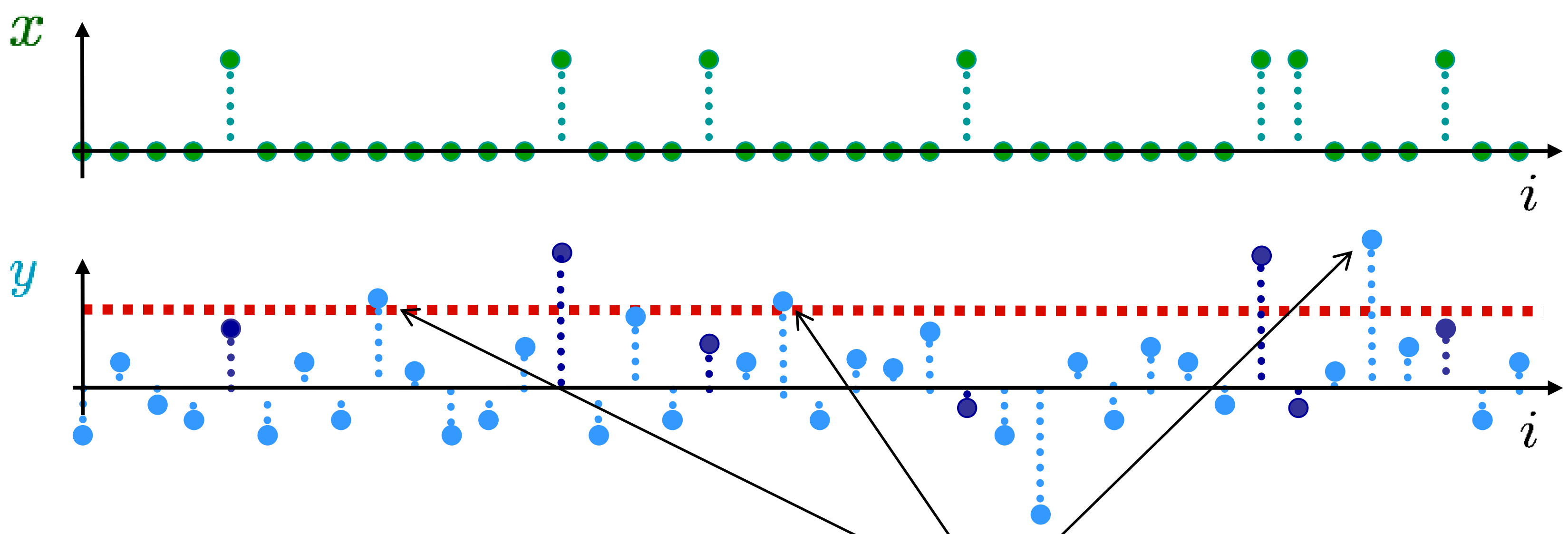
**significant gains when** $s \ll n$

greater sensitivity for same precision budget or lower
experimental requirements for equivalent sensitivity

# *Active Sensing for Sparse Recovery*

## *-- A Relaxed Error Criteria --*

# Measuring Error: False Discoveries



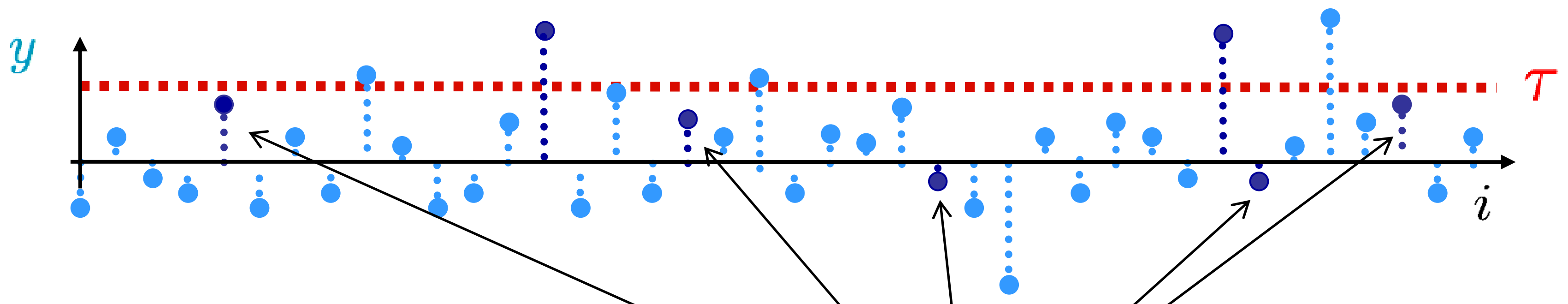

Quantify # of false discoveries (Type I errors)

**Definition:** The *False Discovery Proportion* (FDP) of an estimator $\widehat{\mathcal{S}}$ is

$$\mathrm{FDP}(\widehat{\mathcal{S}}) := \frac{|\widehat{\mathcal{S}} \setminus \mathcal{S}|}{|\widehat{\mathcal{S}}|} = \frac{\#\text{ falsely discovered components}}{\text{total }\#\text{ discovered components}}$$

Here, FDP = **3/5**

# Measuring Error: Non-Discoveries



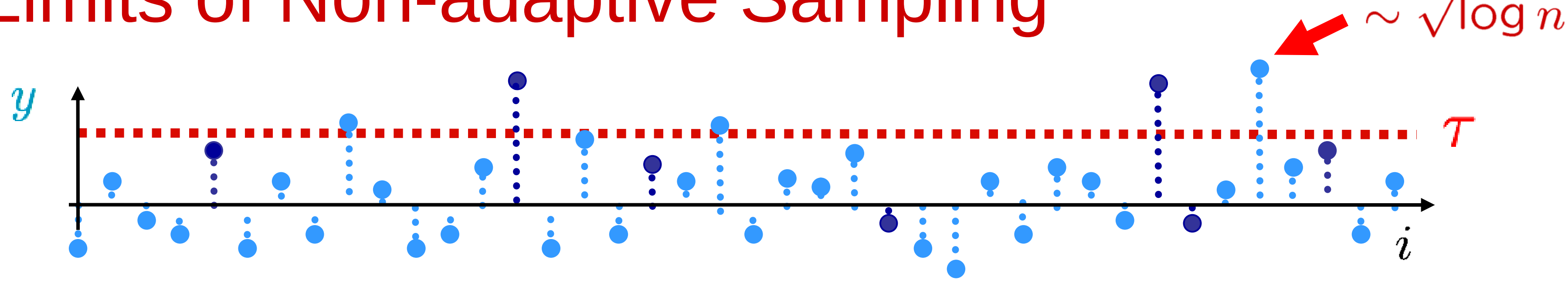

Also interested in quantifying false negatives (Type II errors)

**Definition:** The *Non-Discovery Proportion* (NDP) of $\widehat{\mathcal{S}}$ is

$$\mathrm{NDP}(\widehat{\mathcal{S}}) := \frac{|\mathcal{S} \setminus \widehat{\mathcal{S}}|}{|\mathcal{S}|} = \frac{\#\text{ signal components missed}}{\text{total }\#\text{ signal components}}$$

Here, NDP = **5/7**

# Limits of Non-adaptive Sampling




To determine performance in high-dimensional settings (large $n$), we consider *asymptotic* behavior of FDP and NDP

Assume <u>sublinear</u> sparsity: $|\mathcal{S}| = n^{1-\beta}$ for some fixed $0 < \beta < 1$

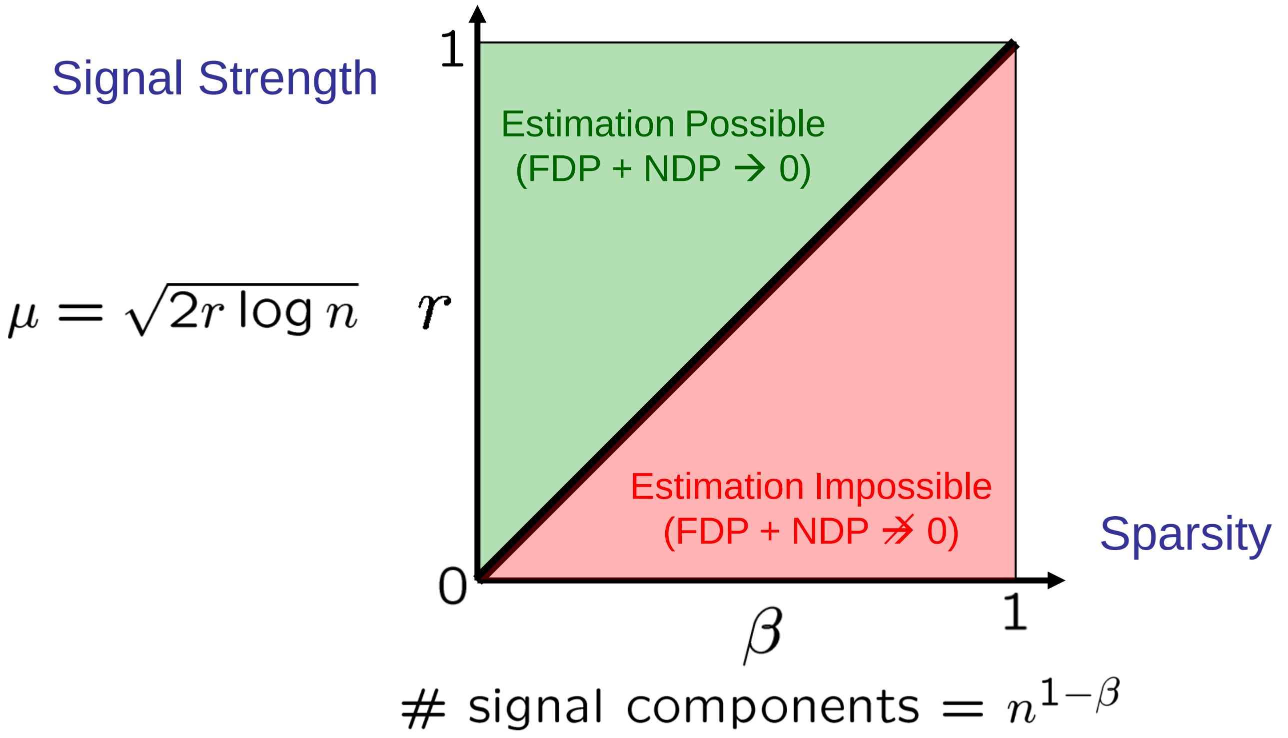
$$\text{e.g.,}\quad \beta = 3/4 \;\Rightarrow\; \begin{array}{ll} n = 10000 & \rightarrow |\mathcal{S}| = 10 \\ n = 1000000 & \rightarrow |\mathcal{S}| = 32 \end{array}$$

**Theorem:** (Donoho & Jin, 2003)
Assume $x$ has $n^{1-\beta}$, $\beta \in (0,1)$, nonzero components of amplitude $\sqrt{2r \log n}$, $r > 0$. If $r > \beta$, there exists a threshold test that yields an estimator $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(y)$ for which

$$\text{FDP}(\widehat{\mathcal{S}}) \xrightarrow{P} 0, \;\; \text{NDP}(\widehat{\mathcal{S}}) \xrightarrow{P} 0, \;\; \text{as } n \rightarrow \infty$$

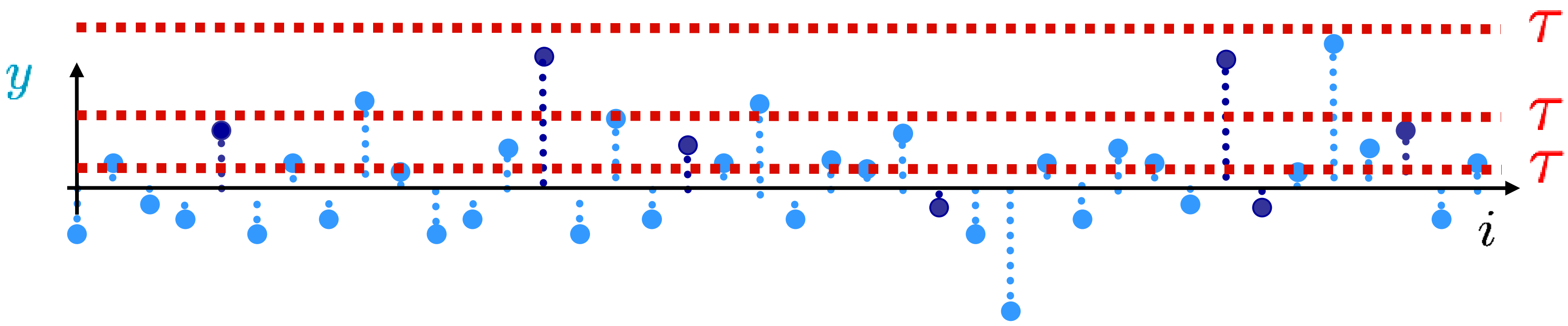
where $\xrightarrow{P}$ denotes convergence in probability. Further, if $r < \beta$, there does not exist a threshold test that can guarantee that both NDP and FDP tend to zero as $n \rightarrow \infty$.

# Sharp Delineation in “Parameter Space”



# What to Do in Low SNR Settings?

What if no signal component amplitudes exceed $\sqrt{2\beta \log n}$?



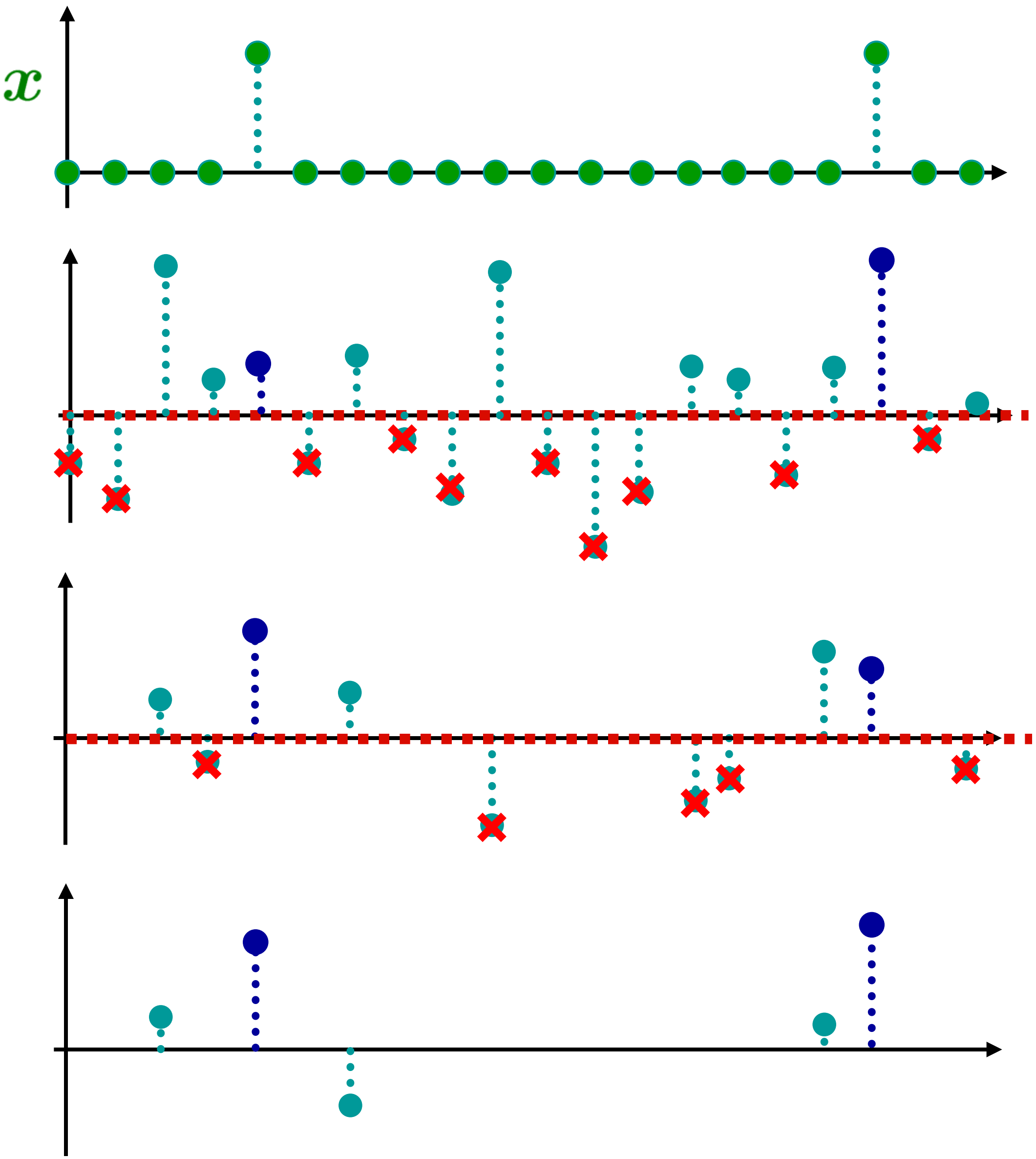

Cannot reliably estimate, but what determinations *can* be made?

**Notice:**

– As $\tau$ decreases, NDP gets closer to 0 (fewer signal components are missed)
– But still *very* unsure which entries are signal components

$\Rightarrow$ Take a closer look at most significant components

# Idealized Example




Let $k = 3$

$R_j = R(n)/3, \;\; j = 1, 2, 3$

$$y_{i,1} = x_i + \mathcal{N}(0, 3)$$

$$y_{i,2} = x_i + \mathcal{N}\left(0, \frac{3}{2}\right)$$

$$y_{i,3} = x_i + \mathcal{N}\left(0, \frac{3}{4}\right)$$

# Distilled Sensing (DS)

*Input*: Number of observation steps: $k$
Precision per step: $R_j$ s.t. $\sum_{j=1}^{k} R_j \leq R(n)$

*Initialize*: Index set $I_1 = \{1, 2, \ldots, n\}$

*Loop*: For each step $j = 1$ to $k$

1) Allocate precision uniformly over $I_j$: $\gamma_{i,j} = R_j/|I_j|, \ i \in I_j$
2) Collect observations $y_{i,j}$ for $i \in I_j$
3) Refinement/distillation: $I_{j+1} = \{i \in I_j : y_{i,j} > 0\}$

*Output*: Final observations: $y_{\text{DS}} := y_{i,k}, \ i \in I_k$

*To recover negative components, replace $y_{i,j}$ by $-y_{i,j}$ in distillation step

**Key Idea:** $|I_{j+1}| \approx |I_j|/2$ when $x$ is sparse

– Assume $R_{j+1}/R_j = \rho > 1/2$, then

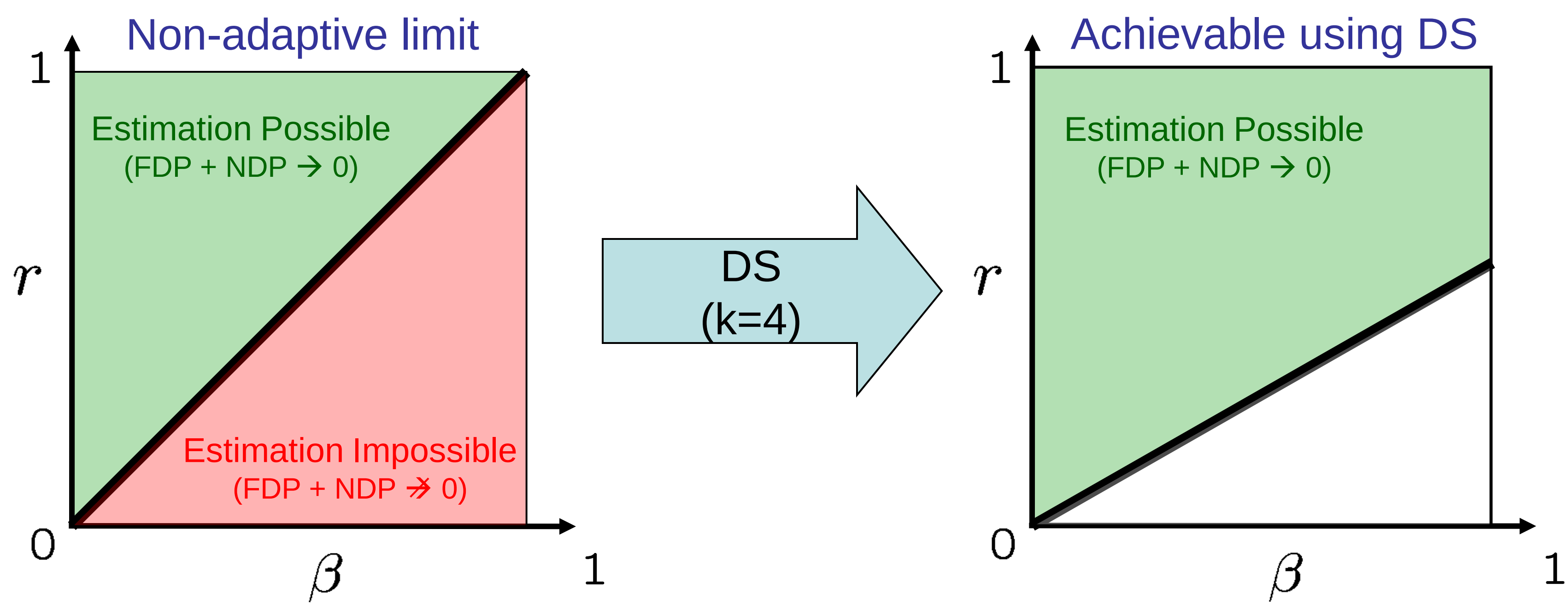
$$\gamma_{i,j+1} = \frac{R_{j+1}}{|I_{j+1}|} \approx 2\rho \frac{R_j}{|I_j|} = 2\rho \ \gamma_{i,j} \ (\text{for } i \in I_j \cap I_{j+1})$$

SNR improvement by $2\rho > 1$

# Equal Allocation of Sensing Resources

**Theorem:** (JH, R. Castro, and R. Nowak, 2008)
Assume $x$ has $n^{1-\beta}$, $\beta \in (0,1)$, nonzero components, and let $R_j = n/k$ (equal precision allocation) for a fixed $k \in \mathbb{N}$. If the signal component amplitudes exceed $\sqrt{2\beta\frac{k}{2^{k-1}}\log n}$, then there exists a threshold test that yields an estimator $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(y_{\mathrm{DS}})$ for which

$$\mathrm{FDP}(\widehat{\mathcal{S}}) \xrightarrow{P} 0, \ \mathrm{NDP}(\widehat{\mathcal{S}}) \xrightarrow{P} 0, \text{ as } n \to \infty.$$



# Equal Allocation of Sensing Resources

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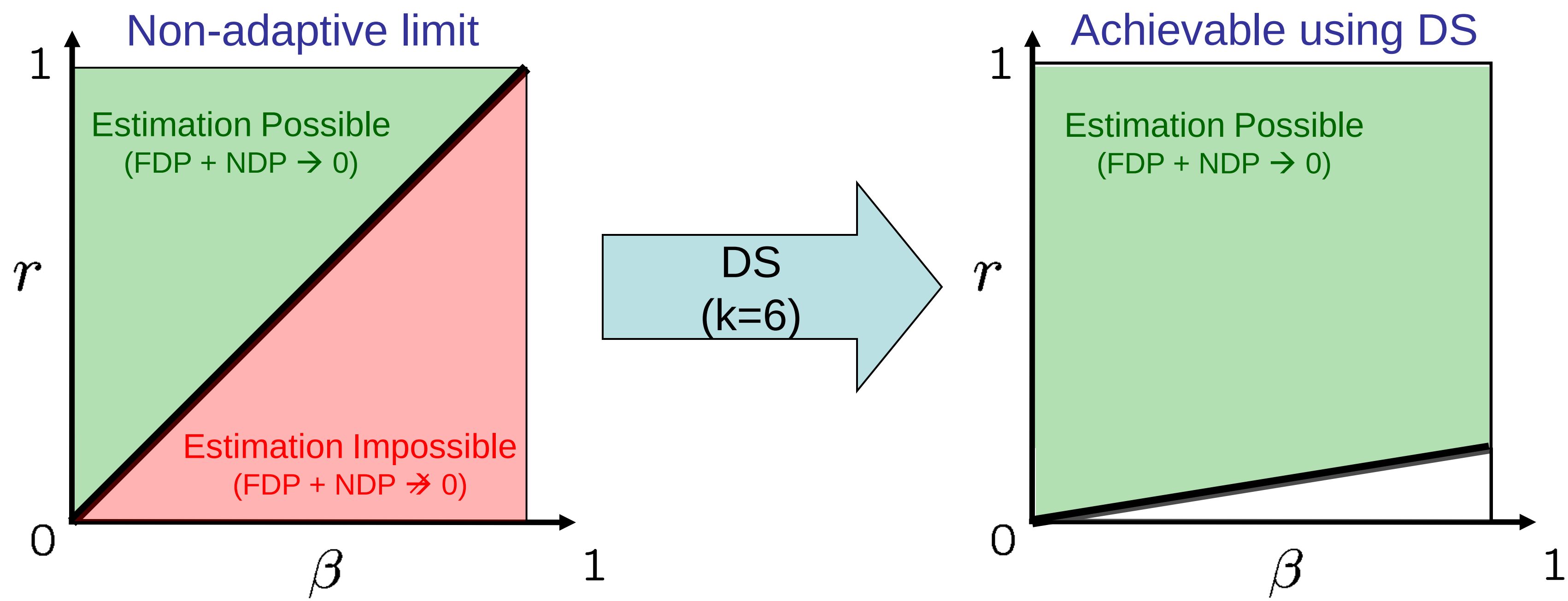
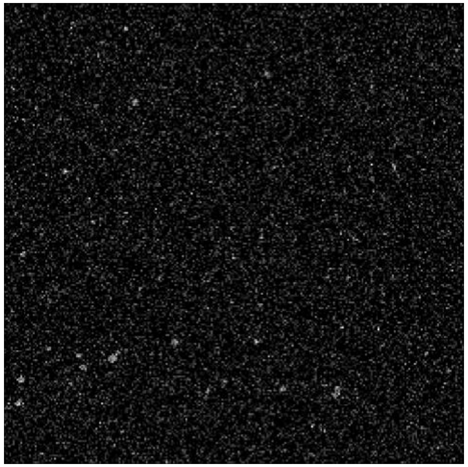
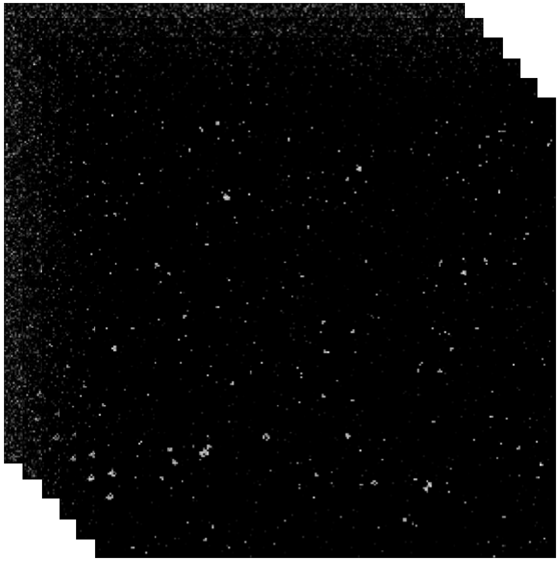


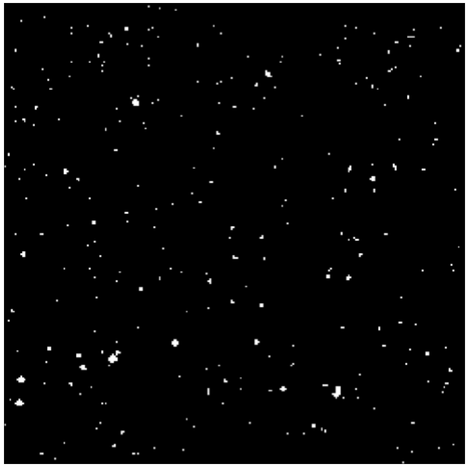
# Recall: Noisy Astronomical Imaging



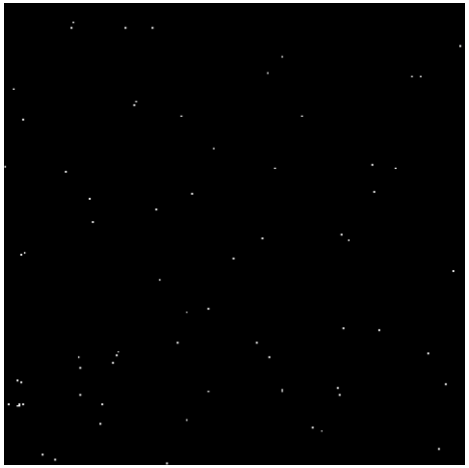
Non-adaptive observations



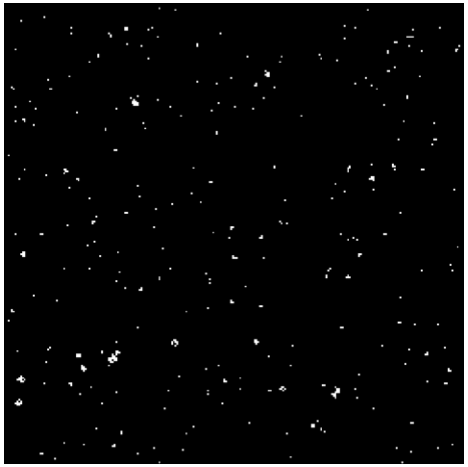
DS observations ($k$=6)



original signal
(~0.8% non-zero components)

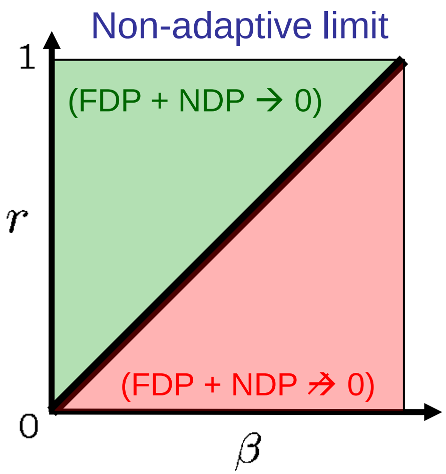


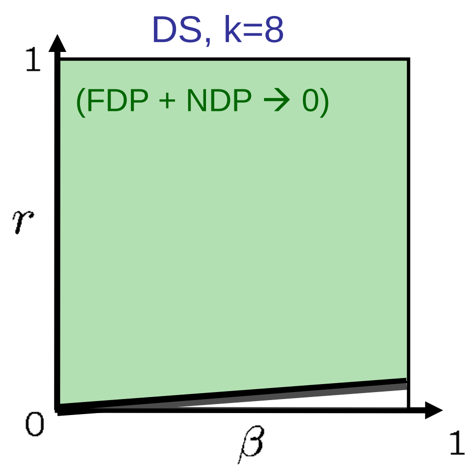
Non-adaptive recovery
(FDP = 0.05)



Adaptive recovery
(FDP = 0.05)

# Unequal Allocation of Sensing Resources…




What about *very weak* signals?
(amplitude $\ll \sqrt{\log n}$)

**Theorem:** (JH, R. Castro, and R. Nowak, 2009)
Assume $x$ has $n^{1-\beta}$, $\beta \in (0,1)$, nonzero components having amplitude at least $\mu(n)$. Choose $k = \lceil \log_2 \log n \rceil + 2$, and precision budget allocated over observation steps such that $\sum_{j=1}^{k} R_j \leq n$, $R_{j+1}/R_j = \rho > 1/2$ for $j = 1, \ldots, k-2$, $R_1 = c_1 n$, and $R_k = c_k n$ for $c_1, c_k \in (0,1)$. From the output of the DS procedure, construct the estimate

$$\widehat{\mathcal{S}}_{\mathrm{DS}} := \left\{ i \in I_k : y_{i,k} > \sqrt{2/c_k} \right\}$$

If $\mu(n)$ is *any* arbitrarily slowly growing function of $n$, then

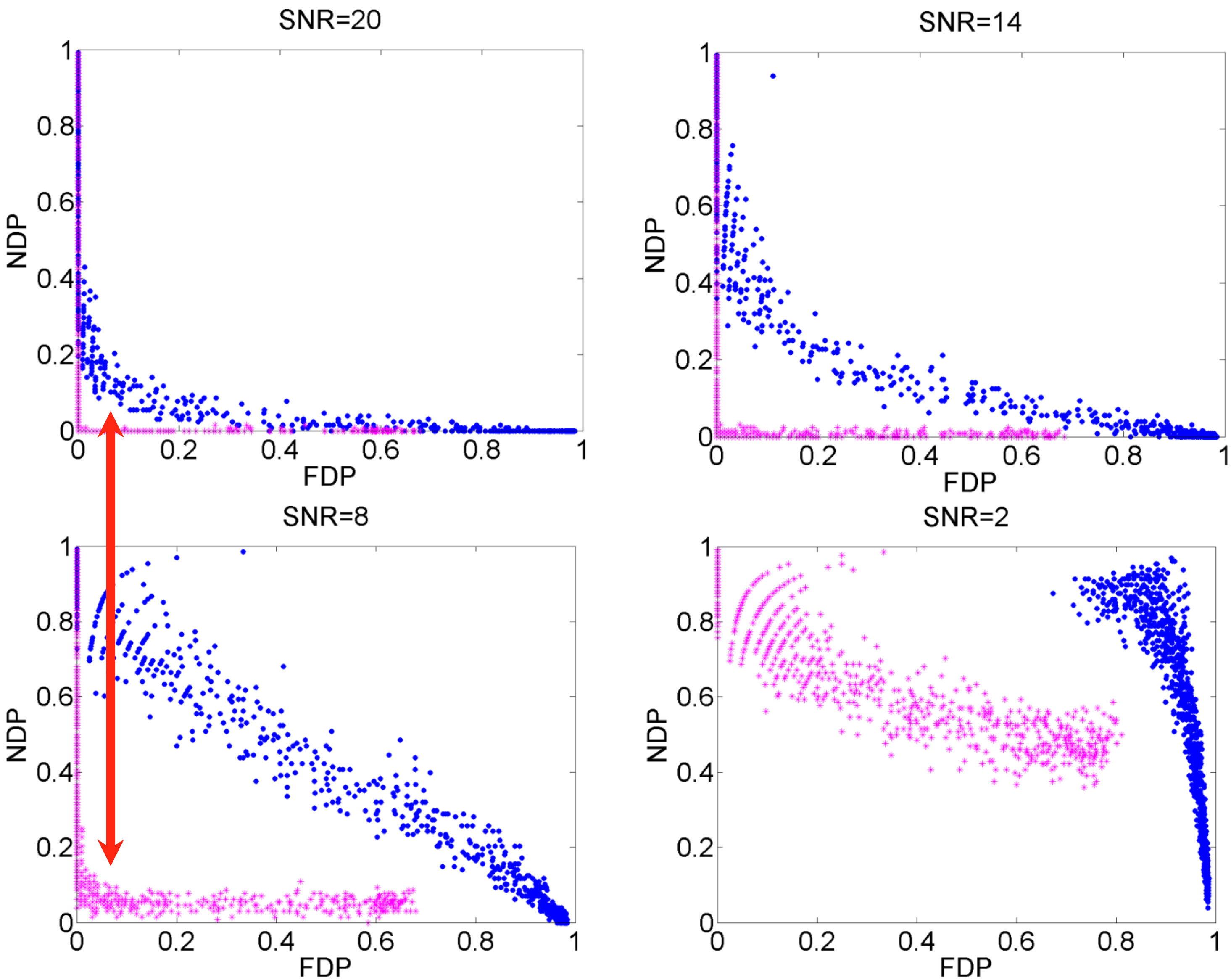
$$\mathrm{FDP}(\widehat{\mathcal{S}}_{\mathrm{DS}}) \xrightarrow{P} 0, \ \mathrm{NDP}(\widehat{\mathcal{S}}_{\mathrm{DS}}) \xrightarrow{P} 0, \ \text{as } n \to \infty.$$

Adaptivity can provide $\sim \log n$ improvement in SNR
and mitigate (or nearly *eliminate*) the curse of dimensionality!

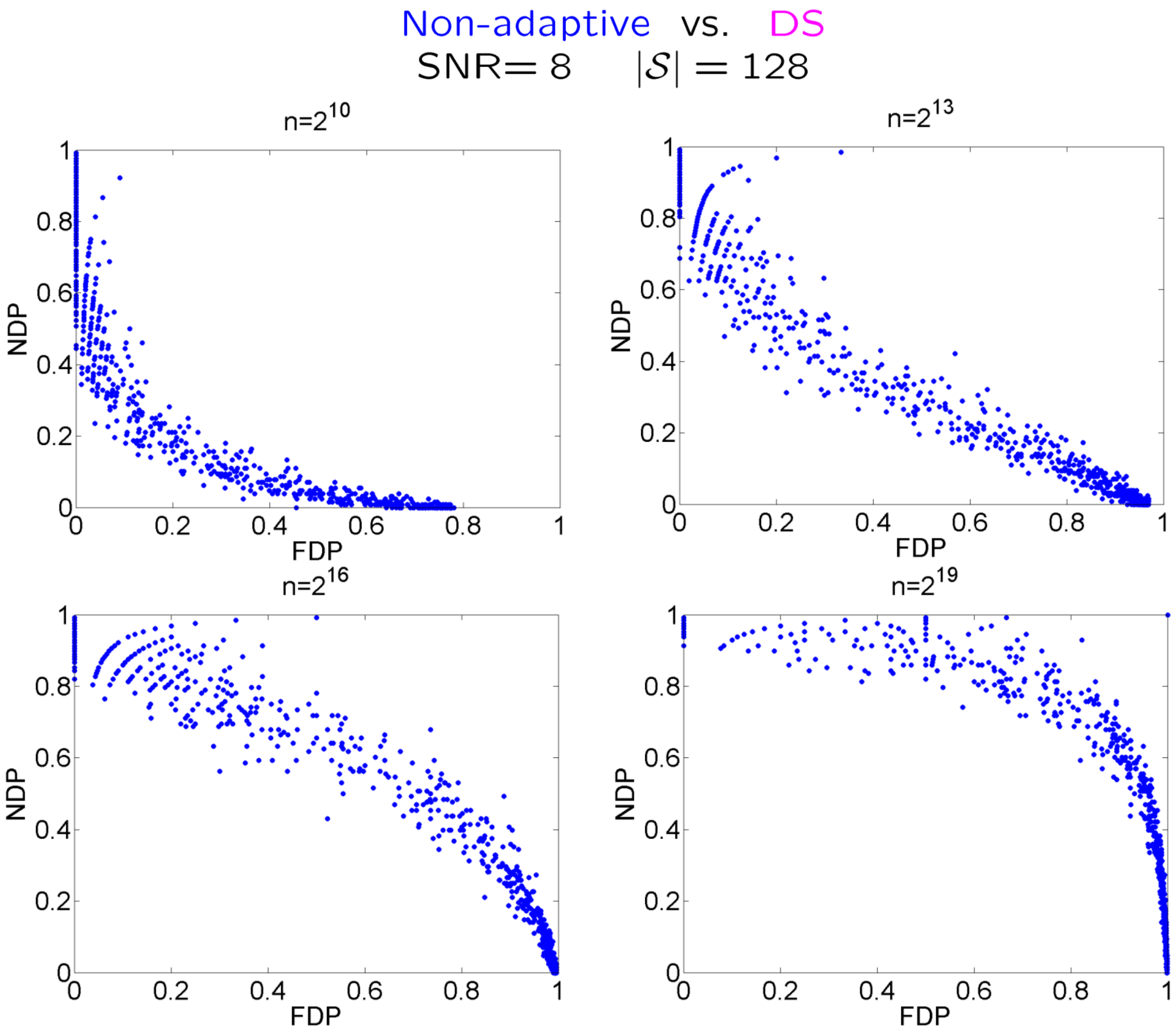
# Simulation

Non-adaptive vs. DS

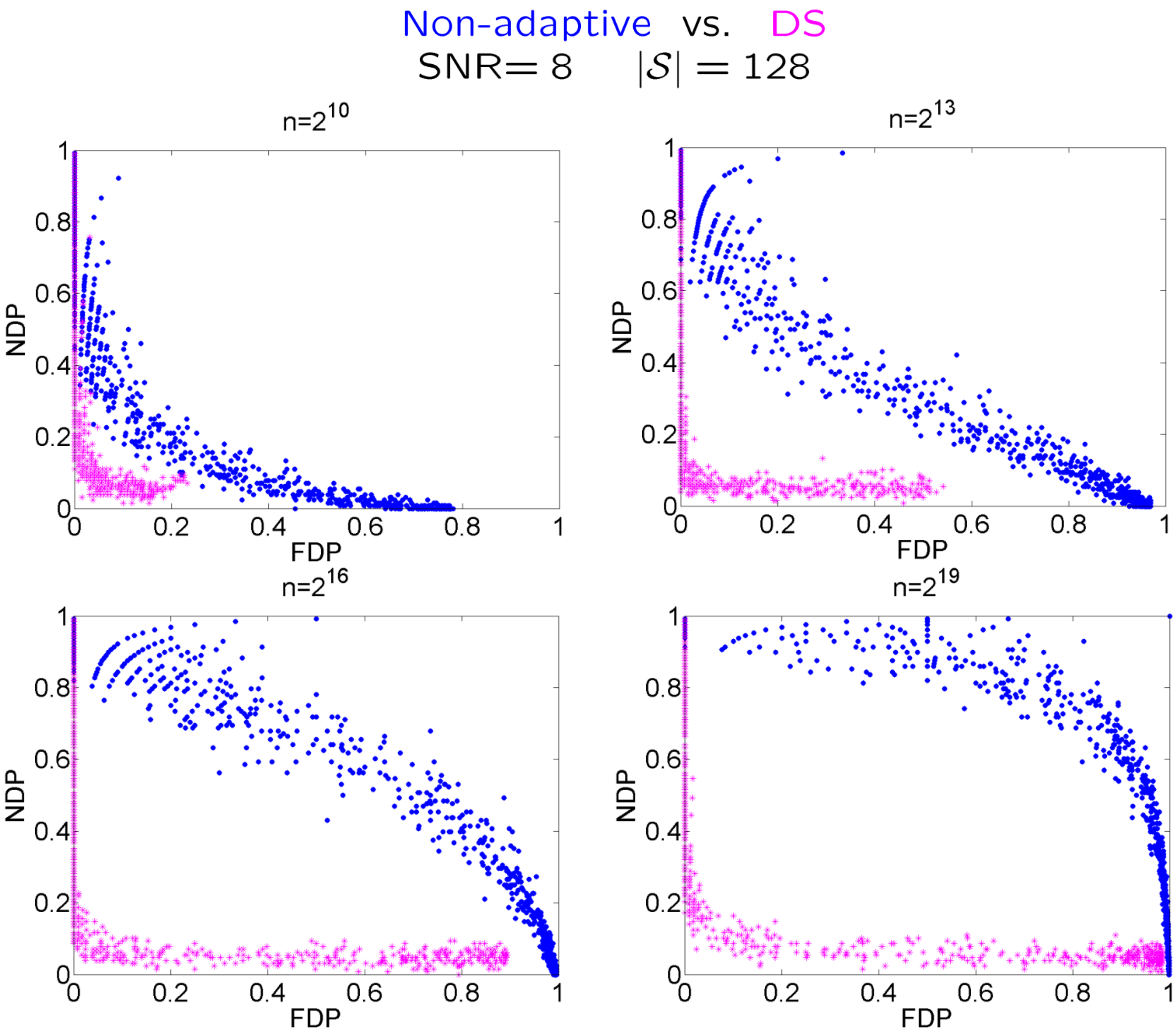
$n = 2^{14} = 16384 \qquad |\mathcal{S}| = \sqrt{n} = 128$



# The Curse of Dimensionality…



# …and the Virtue of Adaptivity!



# *Active Sensing for Sparse Recovery*

## *-- Adaptive Compressive Sampling --*

# Improvements w.r.t. Other Resources?

Note that DS requires about $2n$ total measurements:

- $n$ for first step
- about $n/2$ for second step
- about $n/4$ for third step...

Can we achieve noise-resilience benefits of DS using a reduced # of samples?

## Noisy Compressive Sensing (CS) Observation Model

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} =$$

$$\mathbb{E}\left[\|A\|_F^2\right] = n$$

$$\times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$$

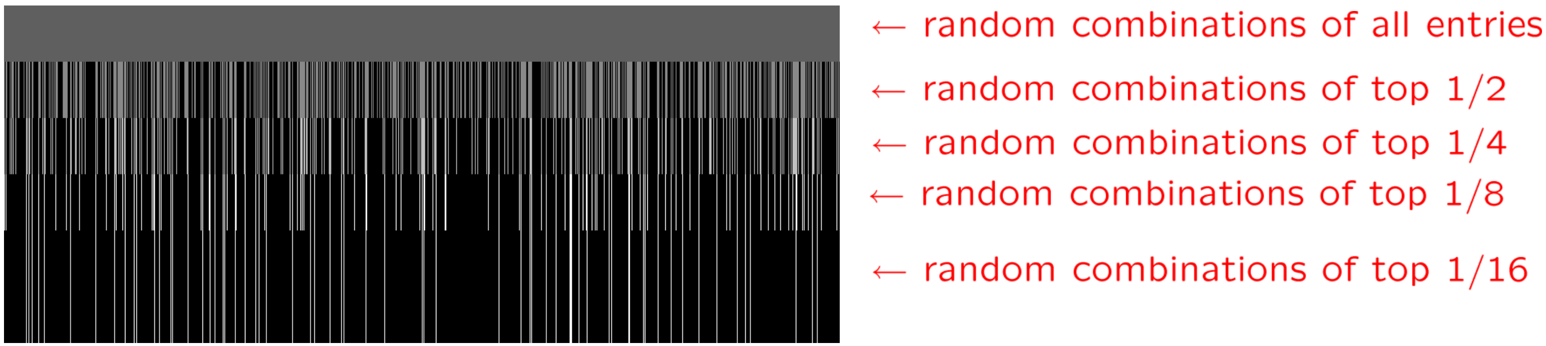
$$z \sim \mathcal{N}(0, I_{m \times m})$$

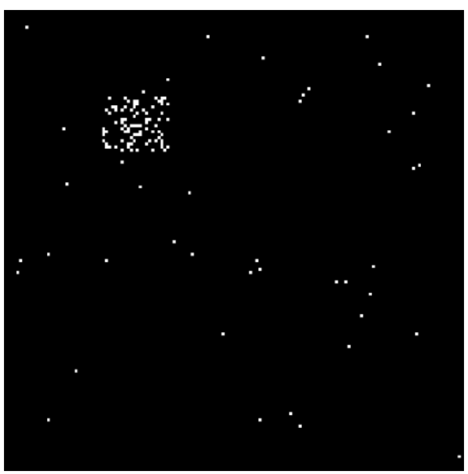
# Compressive DS

**Theorem:** (JH, R. Baraniuk, R. Castro, and R. Nowak, 2009)
Assume $x$ has $|\mathcal{S}| = n^{1-\beta}$ nonzero components of amplitude $\mu$. Collect $O(|\mathcal{S}| \log n)$ adaptive random compressive measurements. When $\mu \succeq \sqrt{\log \log \log n}$, there exists a (tractable) recovery procedure for which

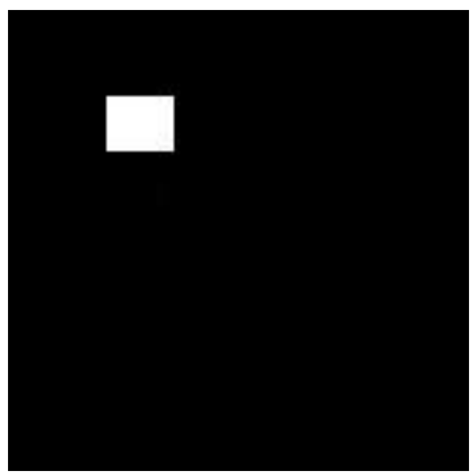
$$\mathrm{FDP}(\widehat{\mathcal{S}}_{\mathrm{CDS}}) \xrightarrow{P} 0, \ \mathrm{NDP}(\widehat{\mathcal{S}}_{\mathrm{CDS}}) \xrightarrow{P} 0, \ \text{as } n \to \infty.$$

Support of random measurement matrix

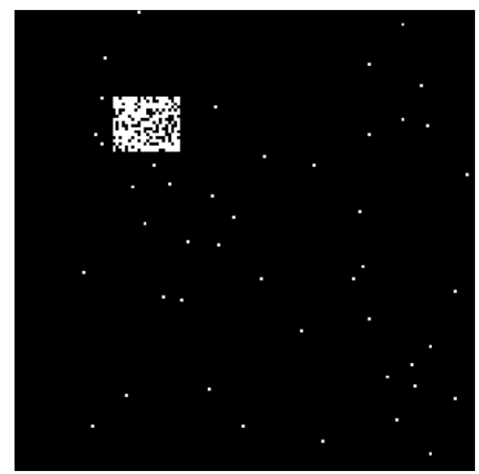


Non-adaptive CS Recovery
(∼ 25% measurements)



Original Signal

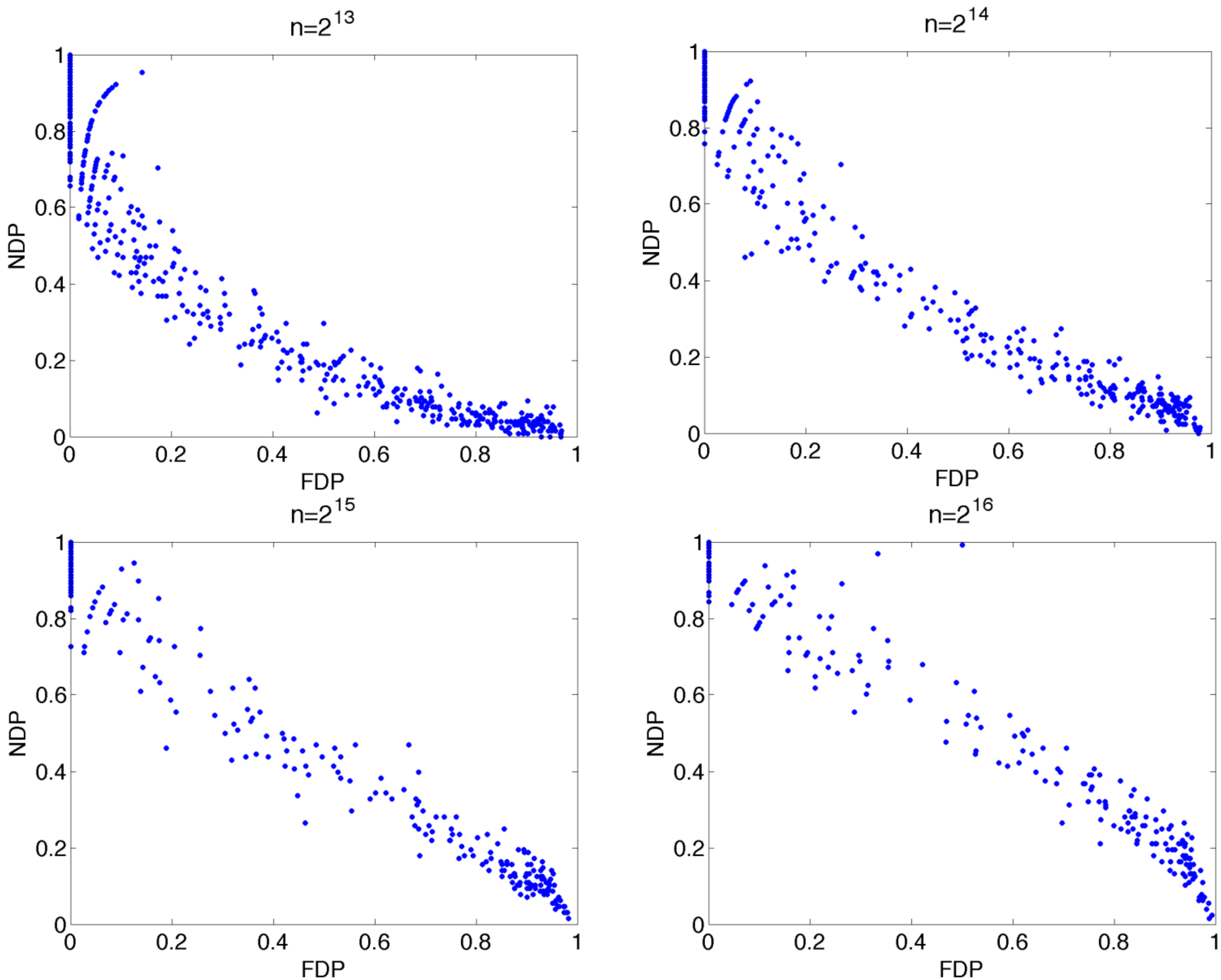


Adaptive CS Recovery
(∼ 25% measurements)

# Curse of Dimensionality: Compressed Sensing

Non-adaptive CS vs. CDS

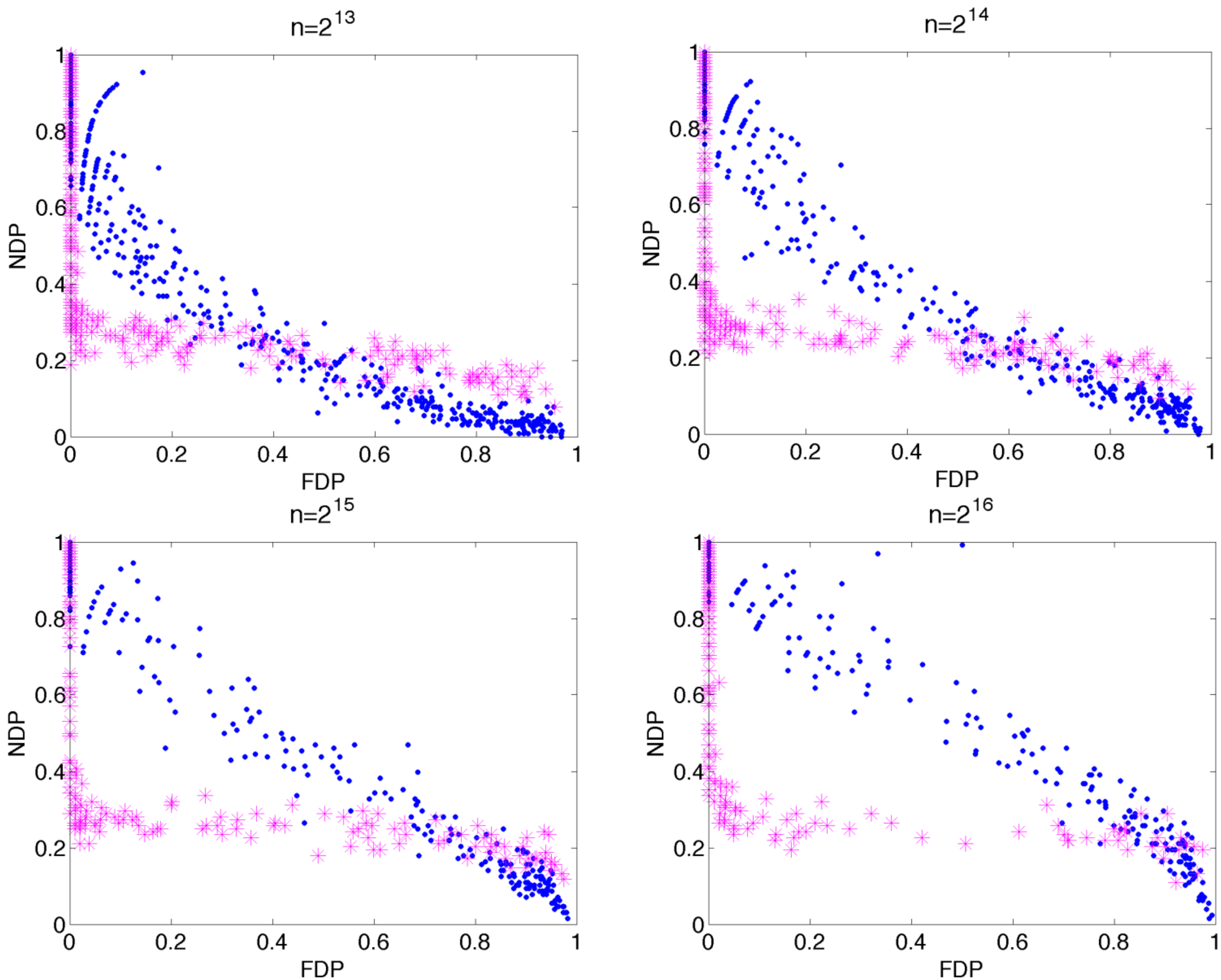
SNR$= 12$ $|\mathcal{S}| = 128$



# …and the Virtue of Adaptivity!

Non-adaptive CS vs. CDS

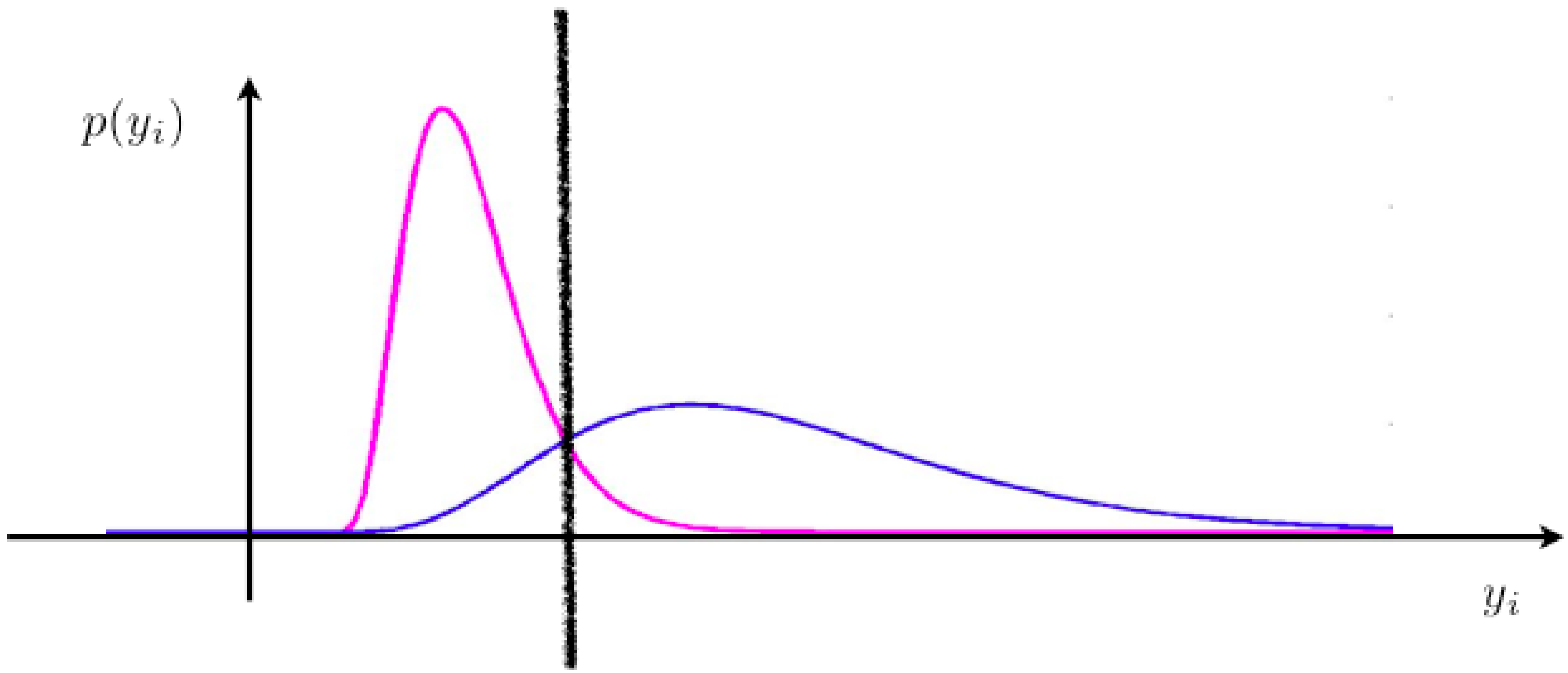
SNR$=12$ $\quad |\mathcal{S}| = 128$



# *Generalization Beyond Gaussian Models*

## *-- Sequential Thresholding --*

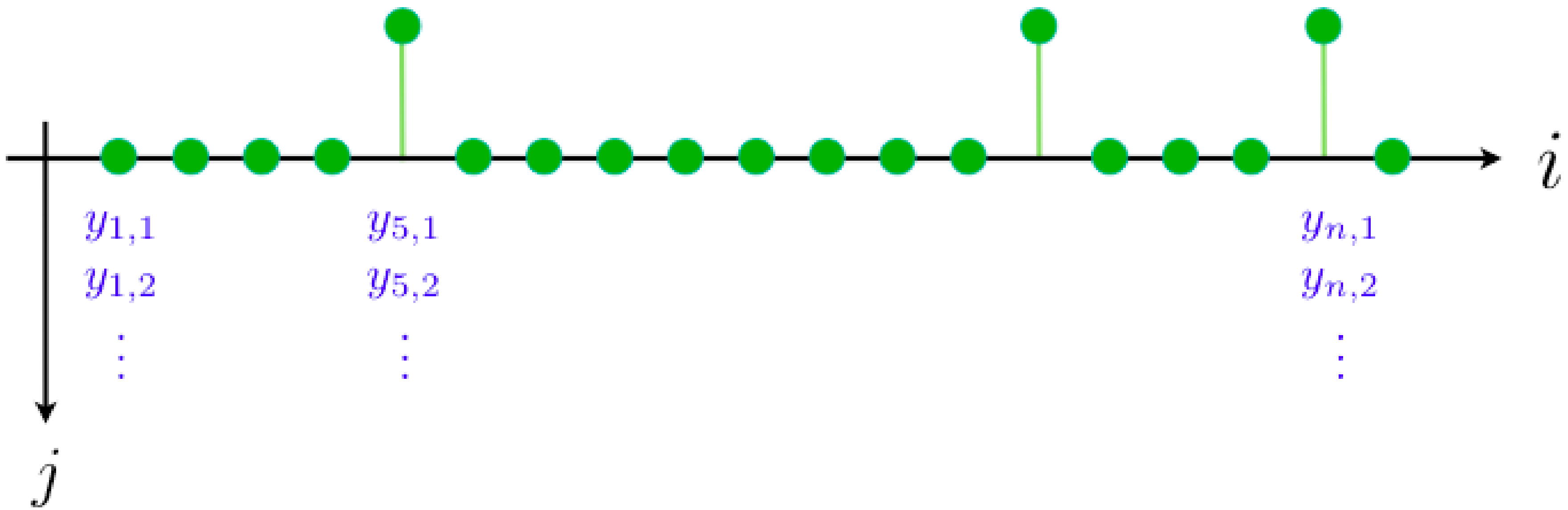
# General Set-up




$$y_i \sim \begin{cases} f_{\theta_0}, & i \notin \mathcal{S} \\ f_{\theta_1}, & i \in \mathcal{S} \end{cases} \qquad i = 1, 2, \ldots, n$$

**Main Assumption**: monotone likelihood ratio

# Multiple Observations, Multiple Tests




$$\{y_{i,j}\}_{j\geq 1} \overset{\text{iid}}{\sim} f_{\theta_0} \quad \text{vs.} \quad \{y_{i,j}\}_{j\geq 1} \overset{\text{iid}}{\sim} f_{\theta_1} \;, \qquad i = 1, 2, \ldots, n$$

# Sequential Thresholding

**sampling/precision budget:** $N = mn \;,\; m \geq 1$

m samples per coordinate on *average*

**log-likelihood ratio statistic:**

$$t_i(m) \;=\; \sum_{j=1}^{m} \log \frac{f_{\theta_1}(y_{i,j})}{f_{\theta_0}(y_{i,j})} \;, \quad i = 1, 2, \ldots, n$$

Collect samples in sets of size $m/2$ and consider sequences of log-LR stats

$$\underbrace{y_{i,1}, \ldots, y_{i,m/2}}_{t_{i,1}(m/2)} \;,\; \underbrace{y_{i,m/2+1}, \ldots, y_{i,m}}_{t_{i,2}(m/2)} \;,\; \cdots$$

**sequential thresholding:** define $\mu_0 := \text{median}\,(t_{i,j}(m/2)\,|\,\theta_0)$

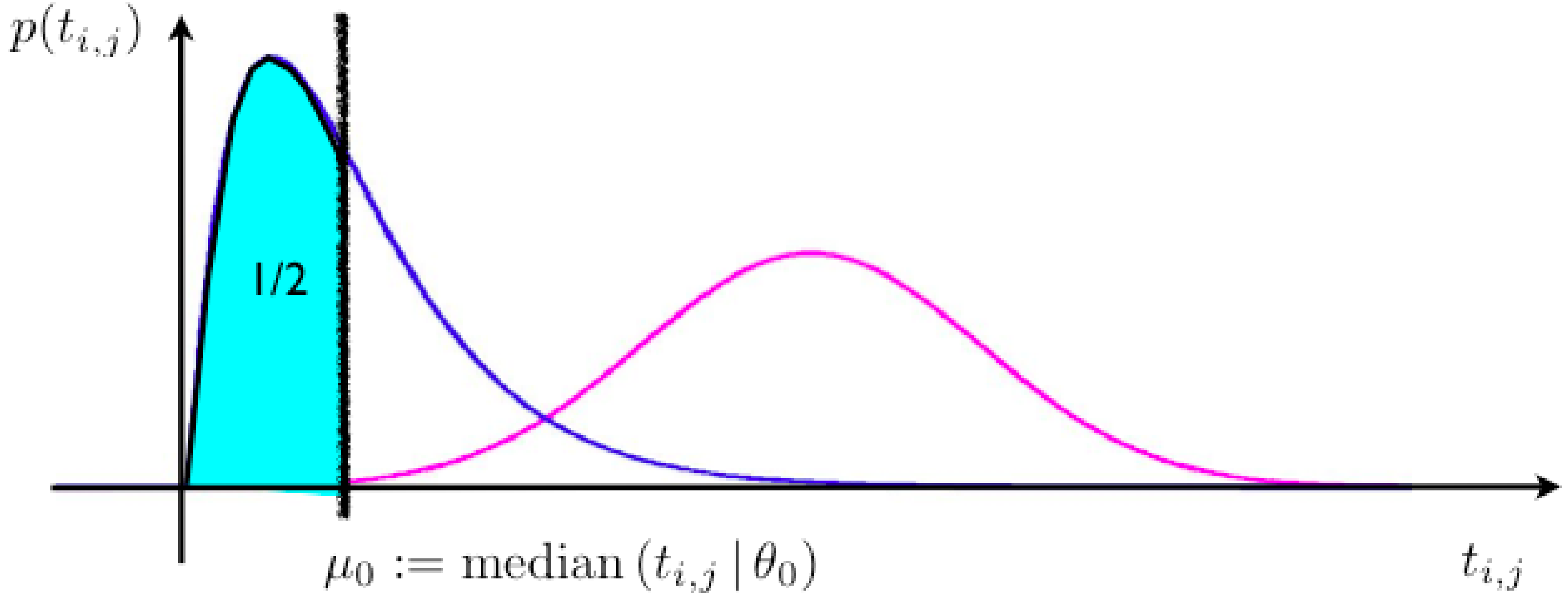
$$t_{i,1}(m/2) \overset{\theta_0}{<} \mu_0 \;\; \text{else} \;\; t_{i,2}(m/2) \overset{\theta_0}{<} \mu_0 \;\; \text{else} \quad \cdots$$

a "cascade" of relatively poor detectors

# Sequential Thresholding

$$\mu_0 := \text{median}\,(t_{i,j}(m/2)\,|\,\theta_0)$$

$$t_{i,1}(m/2) \overset{\theta_0}{<} \mu_0 \;\; \text{else} \;\; t_{i,2}(m/2) \overset{\theta_0}{<} \mu_0 \;\; \text{else} \;\; \cdots$$




if $\{y_{i,j}\} \sim f_{\theta_0}$, then false-positive probability is 1/2 at each step

# Sequential Thresholding

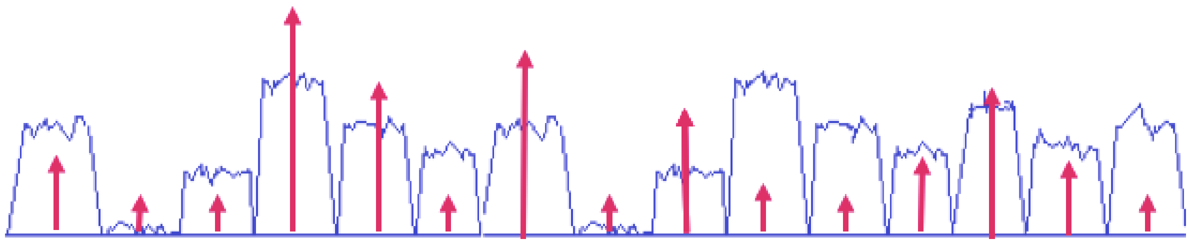
**Sequential Thresholding**

input: $k \approx \log_2 n$ steps, $\mu_0 := \text{median}(t_{i,j}(m/2) \,|\, \theta_0)$
initialize: $\mathcal{S}_0 = \{1, ..., n\}$
**for** $j = 1, \ldots, k$ **do**
  **for** $i \in \mathcal{S}_{j-1}$ **do**
    **measure**: $t_{i,j}$ , $i \in \mathcal{S}_{j-1}$
    **threshold**: $\mathcal{S}_j := \{i \in \mathcal{S}_{j-1} : t_{i,j} > \mu_0\}$
  **end for**
**end for**
output: $\mathcal{S}_k$

total number of samples:

$$\begin{aligned}\mathbb{E}[N] &= \sum_{j=1}^{k} m\mathbb{E}|\mathcal{S}_{j-1}| = m \sum_{j=1}^{k} \left( \frac{n-s}{2^{j-1}} + s \right) \\ &\leq m(n - s + sk) \approx mn \quad (\text{when } s \ll n)\end{aligned}$$

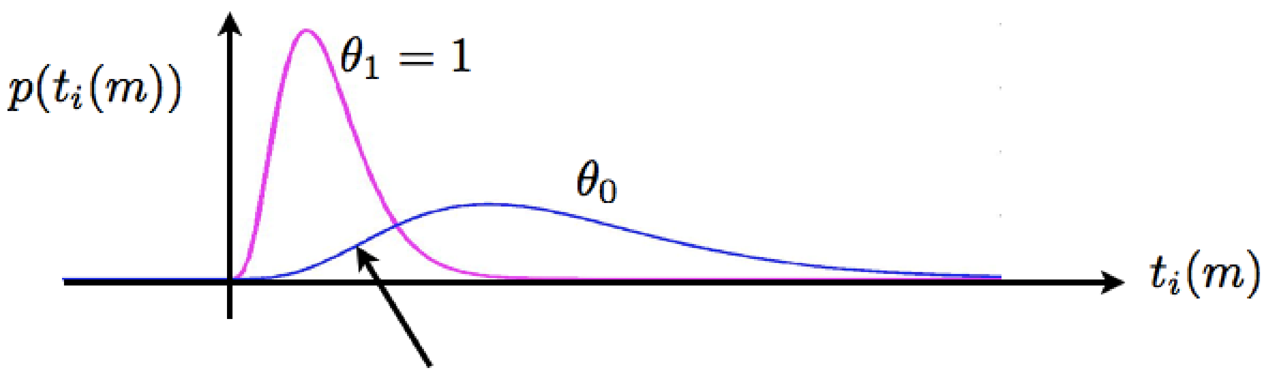
# Spectrum Sensing



goal: find open channel(s) as quickly as possible

**channel samples:** $y_{i,j} \overset{\text{iid}}{\sim} \mathcal{CN}(0, \theta)$ , $\quad \theta_0 > \theta_1 = 1$

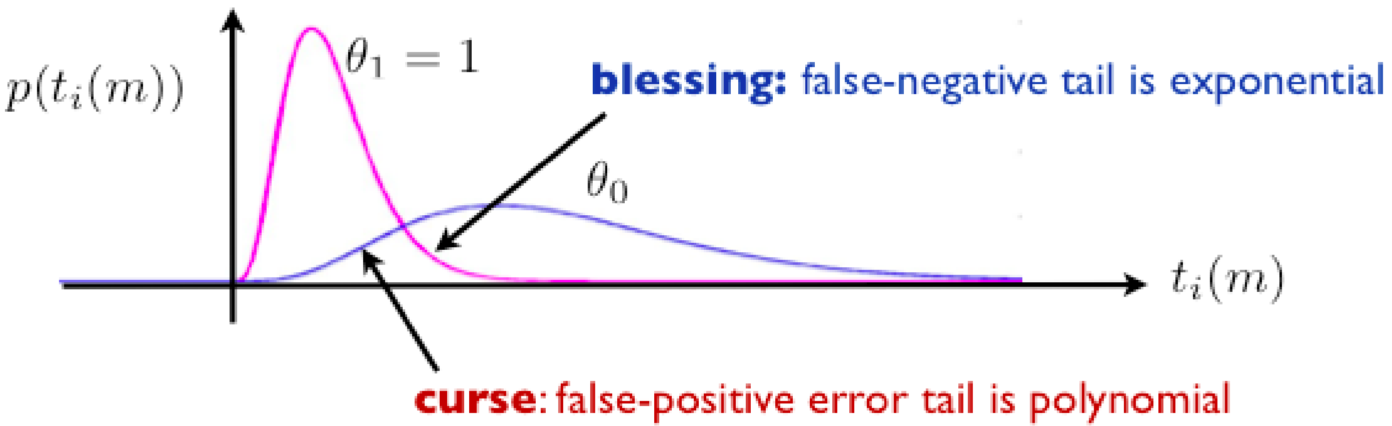
**test statistic:** $t_i(m) = \sum_{j=1}^{m} |y_{i,j}|^2 \sim \begin{cases} \Gamma(m, \theta_0)\,, & i \notin \mathcal{S} \\ \Gamma(m, 1)\,, & i \in \mathcal{S} \end{cases}$




**curse**: false-positive error tail is polynomial

# Spectrum Sensing Application

**test statistic:** $t_i(m) = \sum_{j=1}^{m} |y_{i,j}|^2 \sim \begin{cases} \Gamma(m, \theta_0), & i \notin \mathcal{S} \\ \Gamma(m, 1), & i \in \mathcal{S} \end{cases}$



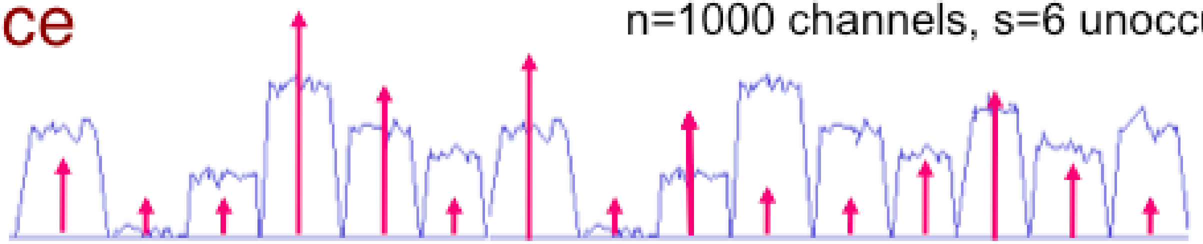

**non-sequential:** $\theta_0 \geq 2(m-1)(n-s)^{1/2m} \sim n^{1/2m}$ (necessary)

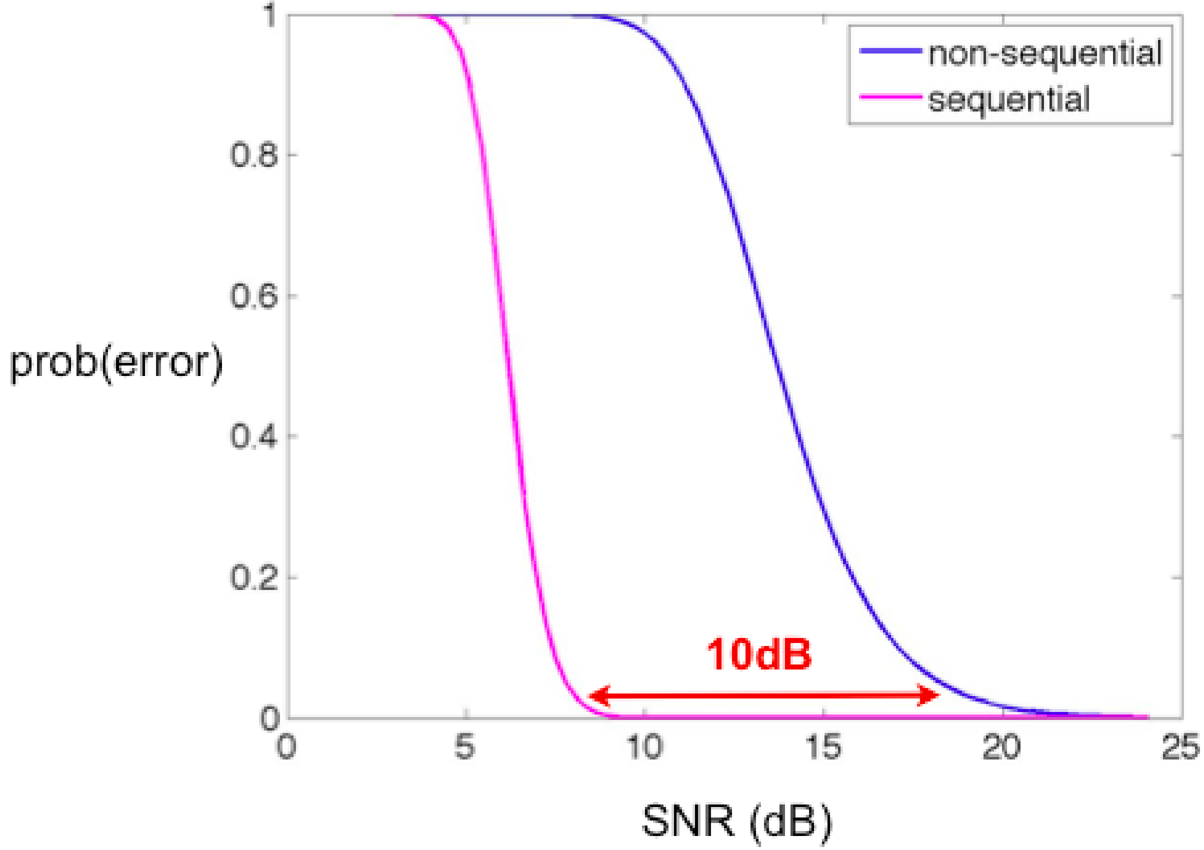
**SPRT:** $\theta_0 \gtrsim \frac{1}{m} \log s$ minimum requirement for any testing scheme with expected sample budget $nm$

**sequential thresholding:** $\theta_0 \geq \frac{1}{2m} \log(s \log_2 n)$ (sufficient)

# Performance

n=1000 channels, s=6 unoccupied






sequential thresholding is about 10 times more sensitive
(for equal scan time) or scans 3-4 times faster (for same reliability)

# End of Presentation…