ECE 830 Fall 2011 Statistical Signal Processing

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Lecture 6: Neyman-Pearson Detectors

In Lecture 5 we saw that the likelihood ratio statistic was optimal for testing between two simple hypotheses. The test simply compares the likelihood ratio to a threshold. The "optimal" threshold is a function of the prior probabilities and the costs assigned to different errors. The choice of costs is subjective and depends on the nature of the problem, but the prior probabilities must be known. Unfortunately, often the prior probabilities are not known precisely, and thus the correct setting for the threshold is unclear.

To deal with this, consider an alternative design specification. Let's design a test that minimizes one type of error subject to a constraint on the other type of error. This constrained optimization criterion does not require knowledge of prior probabilities nor cost assignments. It only requires a specification of the maximum allowable value for one type of error, which is sometimes even more natural than assigning costs to the different errors. A classic result due to Neyman and Pearson shows that the solution to this type of optimization is again a likelihood ratio test.

1 Neyman-Pearson Lemma

Assume that we observe a random variable distributed according to one of two distributions.

$$\begin{aligned} H_0 : X &\sim p_0 \\ H_1 : X &\sim p_1 \end{aligned}$$

In many problems, H_0 is consider to be a sort of baseline or default model and is called the *null hypothesis*. H_1 is a different model and is called the *alternative hypothesis*. If a test chooses H_1 when in fact the data were generated by H_0 the error is called a *false-positive* or *false-alarm*, since we mistakenly accepted the alternative hypothesis. The error of deciding H_0 when H_1 was the correct model is called a *false-negative* or *miss*.

Let T denote a testing procedure based on an observation of X, and let R_T denote the subset of the range of X where the test chooses H_1. The probability of a false-positive is denoted by

$$P_0(R_T) \ := \ \int_{R_T} p_0(x)\, dx \ .$$

The probability of a false-negative is $1 - P_1(R_T)$, where

$$P_1(R_T) \ := \ \int_{R_T} p_1(x)\, dx \ ,$$

is the probability of correctly deciding H_1, often called the *probability of detection*.

Consider likelihood ratio tests of the form

$$\frac{p_1(x)}{p_o(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \ .$$

The subset of the range of X where this test decides H_1 is denoted

$$R_{LR}(\lambda) \ := \ \{x \ : \ p_1(x) > \lambda \, p_0(x)\} \ ,$$

and therefore the probability of a false-positive decision is

$$P_0(R_{LR}(\lambda)) \ := \ \int_{R_{LR}(\lambda)} p_0(x) \, dx \ = \ \int_{\{x : p_1(x) > \lambda p_0(x)\}} p_0(x) \, dx$$

This probability is a function of the threshold λ; the set $R_{LR}(\lambda)$ shrinks/grows as λ increases/decreases. We can select λ to achieve a desired probability of error.

Lemma 1 (Neyman-Pearson)**.** *Consider the likelihood ratio test*

$$\frac{p_1(x)}{p_o(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

with $\lambda > 0$ chosen so that $P_0(R_{LR}(\lambda)) = \alpha$. There does not exist another test T with $P_0(R_T) \leq \alpha$ and $P_1(R_T) > P_1(R_{LR}(\lambda))$. That is, the LRT is the **most powerful test** *with probability of false-positive less than or equal to α.*

Proof. Let T be any test with $P_0(R_T) = \alpha$ and let NP denote the LRT with λ chosen so that $P_0(R_{LR}(\lambda)) = \alpha$. To simplify the notation we will denote use R_{NP} to denote the region $R_{LR}(\lambda)$. For any subset R of the range of X define

$$P_i(R) := \int_R p_i(x) \, dx,$$

This is simply the probability of $X \in R$ under hypothesis H_i. Note that

$$\begin{aligned} P_i(R_{NP}) &= P_i(R_{NP} \cap R_T) + P_i(R_{NP} \cap R_T^c) \\ P_i(R_T) &= P_i(R_{NP} \cap R_T) + P_i(R_{NP}^c \cap R_T) \end{aligned}$$

where the superscript c indicates the complement of the set. By assumption $P_0(R_{NP}) = P_0(R_T) = \alpha$, therefore

$$P_0(R_{NP} \cap R_T^c) = P_0(R_{NP}^c \cap R_T) \ .$$

Now, we want to show

$$P_1(R_{NP}) \geq P_1(R_T)$$

which holds if

$$P_1(R_{NP} \cap R_T^c) \geq P_1(R_{NP}^c \cap R_T) \ .$$

To see that this is indeed the case,

$$
\begin{aligned}
P_1(R_{NP} \cap R_T^c) &= \int_{R_{NP} \cap R_T^c} p_1(x)\,dx \\
&\geq \lambda \int_{R_{NP} \cap R_T^c} p_o(x)\,dx \\
&= \lambda\, P_o(R_{NP} \cap R_T^c) \\
&= \lambda\, P_o(R_{NP}^c \cap R_T) \\
&= \lambda \int_{R_{NP}^c \cap R_T} p_o(x)\,dx \\
&\geq \int_{R_{NP}^c \cap R_T} p_1(x)\,dx \\
&= P_1(R_{NP}^c \cap R_T).
\end{aligned}
$$

□

The probability of a false-positive is also called the probability of false-alarm, which we will denote by P_{FA} in the following examples. We will also denote the probabilty of detection ($1-$ probability of a false-negative) by P_D. The NP test maximizes P_D subject to a constraint on P_{FA}.

1.1 Detecting a DC Signal in Additive White Gaussian Noise

Consider the binary hypotheses

$$
\begin{aligned}
H_0 :\ X_1, \ldots, X_n &\overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \\
H_1 :\ X_1, \ldots, X_n &\overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)\ ,\ \mu > 0
\end{aligned}
$$

and assume that $\sigma^2 > 0$ is known. The first hypothesis is *simple*. It involves a fixed and known distribution. The second hypothesis is simple if μ is known. However, if all we know is that $\mu > 0$, then the second hypothesis is the composite of many alternative distributions, i.e., the collection $\{N(\mu, \sigma^2)\}_{\mu>0}$. In this case, H_1 is called a *composite* hypothesis.

The likelihood ratio test takes the form

$$
\frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(x_i-\mu)^2}}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}x_i^2}} = \frac{\frac{1}{(2\pi\sigma^2)^{n/2}} e^{\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}}{\frac{1}{(2\pi\sigma^2)^{n/2}} e^{\frac{-1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma
$$

The inequalities are preserved if we apply a monotonic transformation to both sides, so we can simplify the expression by taking the logarithm, giving us the log-likelihood ratio test

$$
\frac{-1}{2\sigma^2}\left(-2\mu\sum_{i=1}^n x_i + n\mu^2\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log(\gamma)
$$

Assuming $\mu > 0$, this is equivalent to

$$
\sum_{i=1}^n x_i \underset{H_0}{\overset{H_1}{\gtrless}} \nu\ ,
$$

with $\nu = \frac{\sigma^2}{\mu}\ln\gamma + \frac{n\mu}{2}$, and since γ was ours to choose, we can equivalently choose ν to trade-off between the two types of error. Note that $t := \sum_{i=1}^n x_i$ is simply the sufficient statastic for the mean of a normal distribution. Lets rewrite our hypotheses in terms of the sufficient statistic:

$$H_0 : t \sim N(0, n\sigma^2) \qquad H_1 : t \sim N(n\mu, n\sigma^2)$$

Let's now determine P_{FA} and P_D for the log-likelihood ratio test.

$$P_{FA} \;=\; \int_\nu^\infty \frac{1}{\sqrt{2n\pi\sigma^2}} e^{-\frac{t^2}{2n\sigma^2}}\, dt \;=\; Q\left(\frac{\nu}{\sqrt{n\sigma^2}}\right) \;,$$

where $Q(z) = \int_{u\geq z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$, the tail probability of the standard normal distribution. Similarly,

$$P_D \;=\; \int_\nu^\infty \frac{1}{\sqrt{2n\pi\sigma^2}} e^{-\frac{(t-n\mu)^2}{2n\sigma^2}}\, dt \;=\; Q\left(\frac{\nu - n\mu}{\sqrt{n\sigma^2}}\right) \;.$$

In both cases the expression in terms of the Q function is the result of a simple change of variables in the integration. The Q function is invertible, so we can solve for the value of ν in terms of P_{FA}, that is $\nu = \sqrt{n\sigma^2}Q^{-1}(P_{FA})$. Using this we can express P_D as

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{n\mu^2}{\sigma^2}}\right),$$

where $\sqrt{\frac{n\mu^2}{\sigma^2}}$ is simply the signal-to-noise ratio (SNR). Since $Q(z) \to 1$ as $z \to -\infty$, it is easy to see that the probability of detection increases as μ and/or n increase.

1.2 Detecting a Change in Variance

Consider the binary hypotheses

$$\begin{aligned} H_0 :\; & X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(0, \sigma_0^2) \\ H_1 :\; & X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(0, \sigma_1^2) \;,\; \sigma_1 > \sigma_0 \end{aligned}$$

The log-likelihood ratio test is

$$\frac{n}{2}\log\left(\frac{\sigma_0^2}{\sigma_1^2}\right) + \left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_{i=1}^n x_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma) \;.$$

Some simple algebra shows

$$\sum_{i=1}^n x_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} \nu$$

with $\nu = 2\left(\frac{\sigma_1^2\sigma_o^2}{\sigma_1^2 - \sigma_0^2}\right)(\log(\gamma) + n\ln(\frac{\sigma_1}{\sigma_o}))$. Note that $t := \sum_{i=1}^n x_i^2$ is the sufficient statistic for variance of a zero-mean normal distribution.

Now recall that if $X_1, \ldots, , X_n \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^n X_i^2 \sim \chi_n^2$ (chi-square distributed with n degrees of freedom). Let's rewrite our null hypothesis test using the sufficient statistic:

$$H_0 \ : \ t = \sum_{i=1}^n \frac{x_i^2}{\sigma_0^2} \sim \chi_n^2$$

The probability of false alarm is just the probability that a χ_n^2 random variable exceeds ν/σ_0^2. This can be easily computed numerically. For example, if we have $n = 20$ and set $P_{FA} = 0.01$, then the correct threshold is $\nu = 37.57\sigma_0^2$.