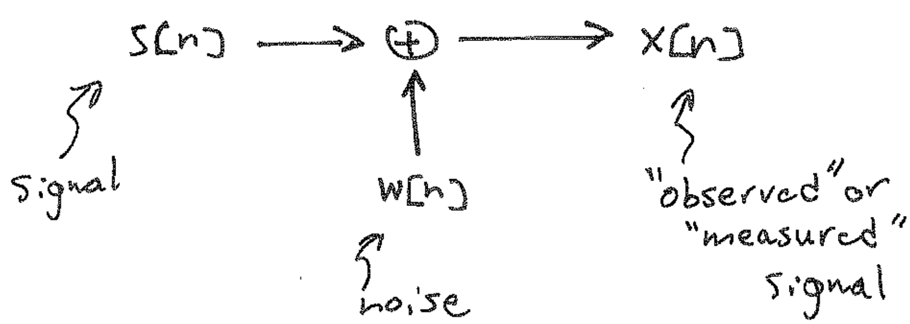
Optimal Filter Design from a Statistical Viewpoint

Basic Problem:



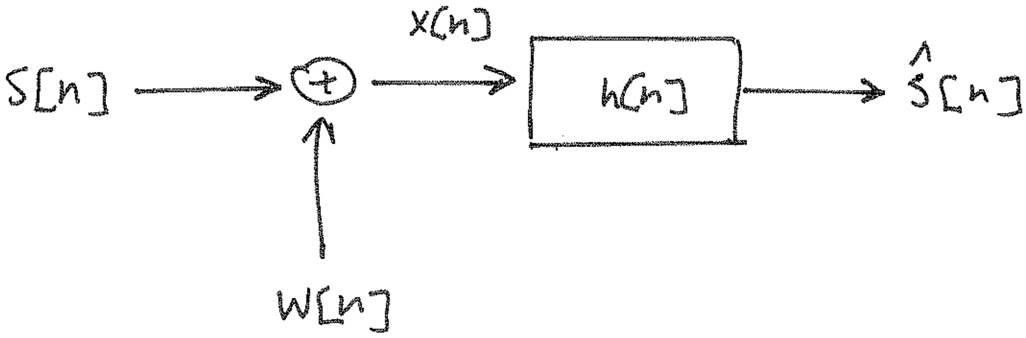
Can we recover s[n] from x[n]?

Applications:

- Comm systems
- control systems
- geophysics
- image processing
- speech enhancement

Goal:

Design a linear time-invariant filter to "reduce" the noise or "estimate" the signal.
(equivalently)



We want to design $h[n]$ so that $\hat{s}[n]$ is as close as possible to $s[n]$.

Partial Knowledge:

If we know nothing at all about the signal or noise, then we have no basis on which to begin.

On the other hand, if we know the signal or noise exactly, then the problem is trivial.

Ex. If we know $w[n]$ (for example, if we could somehow measure it alone), then we have

$$s[n] = x[n] - w[n]$$

A reasonable and workable compromise between the two extremes above is to assume some partial knowledge of the signal and/or noise characteristics.

Here, we will model the signal and noise as realizations of stationary random processes. Moreover, we will assume:

1. $s[n]$ and $w[n]$ are statistically independent

2. the autocorrelation functions $R_{ss}[n]$ and $R_{ww}[n]$ are known

3. both $s[n]$ and $w[n]$ are zero-mean processes

<u>Remark 1</u>: It is not unreasonable to suppose that the signal and noise arise from separate and unrelated physical mechanisms. This supports assumption 1.

Remark 2: Knowledge of the autocorrelation functions is equ valent to knowing the power spectral densities:

$$S_{ss}(\omega) \overset{\text{DTFT}}{\longleftrightarrow} R_{ss}[n]$$

Thus, assumption 2 is essentially saying that we know how the signal or noise energy is distributed in frequency (on average).

Remark 3: Knowledge of the autocorrelation functions of power spectra may be based on a priori physical models or derived from previous measurements of similar signals.

Ex. Suppose we observe many realizations of the noise, say $w_1[n], \ldots, w_M[n]$. Then

$$R_{ww}[k] \approx \frac{1}{M} \sum_{i=1}^{M} w_i[n] w_i[n+k]$$

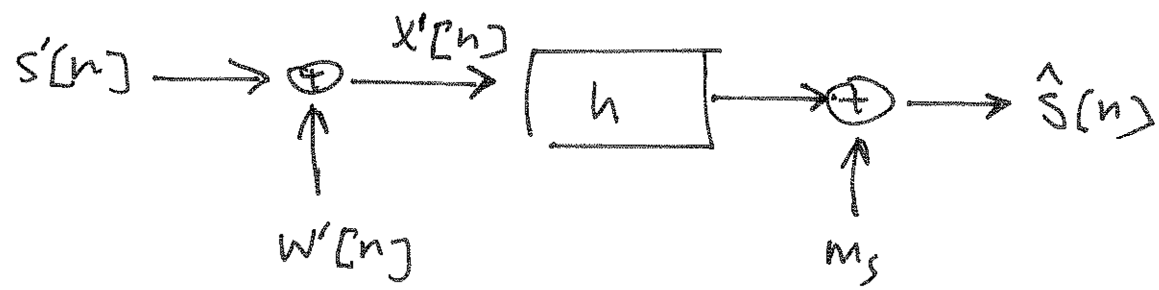
Remark 4: If the signal and/or noise are not zero-mean, but have known mean values, the we can simply redefine them:

$$s'[n] = s[n] - m_s$$

$$w'[n] = w[n] - m_w$$

zero-mean signals

Then, with $x'[n] = x[n] - m_s - m_w$, we have



So, as long as the means are known, without loss of generality we may assume that $s[n]$ and $w[n]$ are zero-mean.

Optimality Criterion:

Based on the partial knowledge we have assumed, we are now in a position to quantify the performance of a given filter $h[n]$. The partial knowledge is based on the "average" signal and noise characteristics, so it is natural to measure the average or expected error of the filter.

$$\left.\begin{aligned} e[n] &= s[n] - \hat{s}[n] \\ &= s[n] - h[n] * x[n] \end{aligned}\right\} \text{error}$$

Mean-square error:

$$E\left[e^2[n]\right] \quad , \text{ average square error}$$

The MSE is a function of $h[n]$:

$$MSE(h) = E\left[\left(s[n] - h[n] * x[n]\right)^2\right]$$

Minimizing the MSE:

The optimum lineare filter, in the sense of minimum mean-square error (MMSE), is called the Wiener filter.

Nobert Wiener

Born: 1894 in Columbia, missouri

Received: Ph.D. math from Harvard at age 18.

with MIT, 1919 –

Died: 1964

Let's look more closely at the MSE.

$$MSE(h) = E\left[\left(s[n] - h[n] * x[n]\right)^2\right]$$

$$= E\left[\left(s[n] - \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right)^2\right]$$

$$= E\left[s^2[n] - 2\sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \cdot s[n] + \left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right)^2 \right]$$

Note that MSE(h) is quadratic in each h[m]. Therefore, MSE(h) has a unique minimum at

$$0 = \frac{\partial}{\partial h[m]} E\left[s^2[n] - 2\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\, s[n] + \left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right)^2\right]$$

$$= E\left[\frac{\partial s^2[n]}{\partial h[m]} - 2\sum_{k=-\infty}^{\infty} \frac{\partial h[k]\, x[n-k]\, s[n]}{\partial h[m]} + \frac{\partial}{\partial h[m]}\left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right)^2 \right]$$

$$= E\Big[0 - 2\,x[n-m]\,s[n]$$

$$+ 2\Big(\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\Big)\cdot x[n-m] \Big]$$

$$= -2R_{xs}[m] + 2\sum_{k=-\infty}^{\infty} h[k]\,R_{xx}[m-k]$$

or

$$\boxed{\sum_{k=-\infty}^{\infty} h[k]\,R_{xx}[m-k] = R_{xs}[m]}$$

The MMSE-optimal filter satisfies the above system of equations known as the <u>Wiener-Hopf Equation</u>.

Note that assumptions 1 and 2 imply

$$R_{xx}[\ell] = R_{ss}[\ell] + R_{ww}[\ell]$$

$$R_{xs}[\ell] = R_{ss}[\ell]$$

are known sequences.

Resulting MMSE:

$h[n]$ satisfies

$$\sum_{k=-\infty}^{\infty} h[k]\, R_{xx}[m-k] = R_{xs}[m]$$

$$\begin{aligned}
MSE(h) &= E\Big[\, s^2[n] - 2 \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\, s[n] + \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} h[k]\, h[\ell] \\
&= R_{ss}[0] - 2 \sum_{k=-\infty}^{\infty} h[k]\, R_{xs}[k] \\
&\qquad + \underbrace{\sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} h[k]\, h[\ell]\, R_{xx}[k-\ell]}_{\sum_k h[k] \underbrace{\sum_\ell h[\ell]\, R_{xx}[k-\ell]}_{R_{xs}[k]}} \\
&= R_{ss}[0] - \sum_{k=-\infty}^{\infty} h[k]\, R_{xs}[k]
\end{aligned}$$

Orthogonality Principle:

Intuitively, we expect that the Wiener filter (mmse filter) "extracts" the maximal amount of signal information from the noisy observation.

This intuition is supported by the so-called <u>orthogonality principle</u>, which states that the error of the Wiener filter, $s[n] - h[n] * x[n]$, is orthogonal to the measurement $x[n]$.

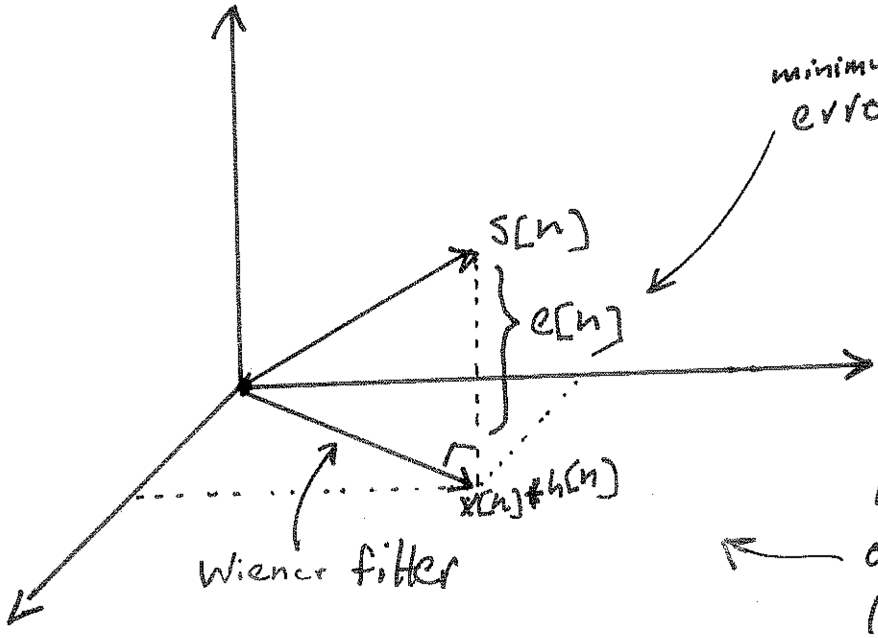
Indeed, for every $m \in \mathbb{Z}$ we have

$$E\left[\left(s[n] - \sum_k h[k]\, x[n-k]\right) \cdot x[n-m]\right]$$

$$= R_{xs}[m] - \sum_k h[k]\, R_{xx}[m-k]$$

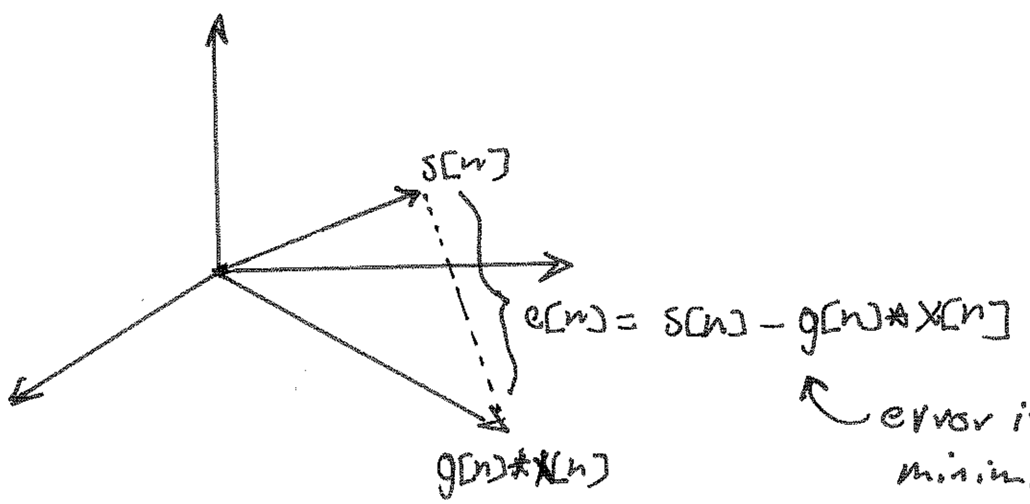
$$= 0$$

minimum error is "statistically" orthogonal to plane



range of all possible linear filters applied to x[n] in this plane

– suboptimal filter



error is not minimized

Frequency Domain Interpretation

MMSE/Wiener filter

$$\sum_{k=-\infty}^{\infty} h[k]\, R_{XX}[m-k] = R_{XS}[m]$$

or equivalently

$$\sum_{k=-\infty}^{\infty} h[k]\left(R_{SS}[m-k] + R_{WW}[m-k]\right) = R_{SS}[m]$$

convolution

Take DTFT of both sides:

$$H(\omega)\left(S_{SS}(\omega) + S_{WW}(\omega)\right) = S_{SS}(\omega)$$

$$\Rightarrow \quad \boxed{H(\omega) = \frac{S_{SS}(\omega)}{S_{SS}(\omega) + S_{WW}(\omega)}}$$

$$H(\omega) = \frac{\text{signal power @ } \omega}{\text{signal + noise power @ } \omega}$$

$$S_{ss}(\omega) >> S_{ww}(\omega)$$

$$\Rightarrow \quad H(\omega) \approx 1$$

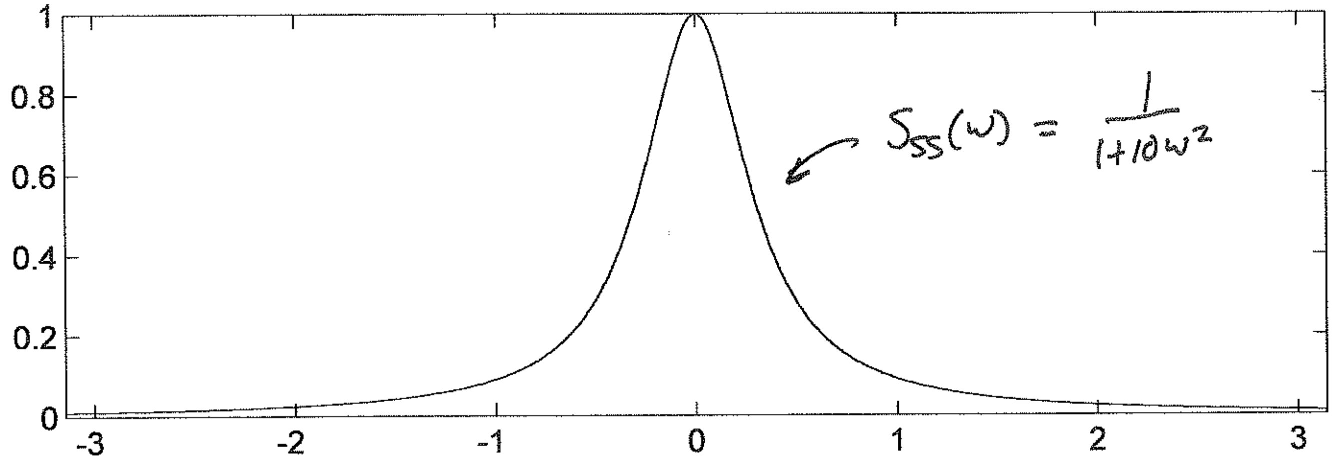
$$S_{ss}(\omega) << S_{ww}(\omega)$$

$$\Rightarrow \quad H(\omega) \approx 0$$

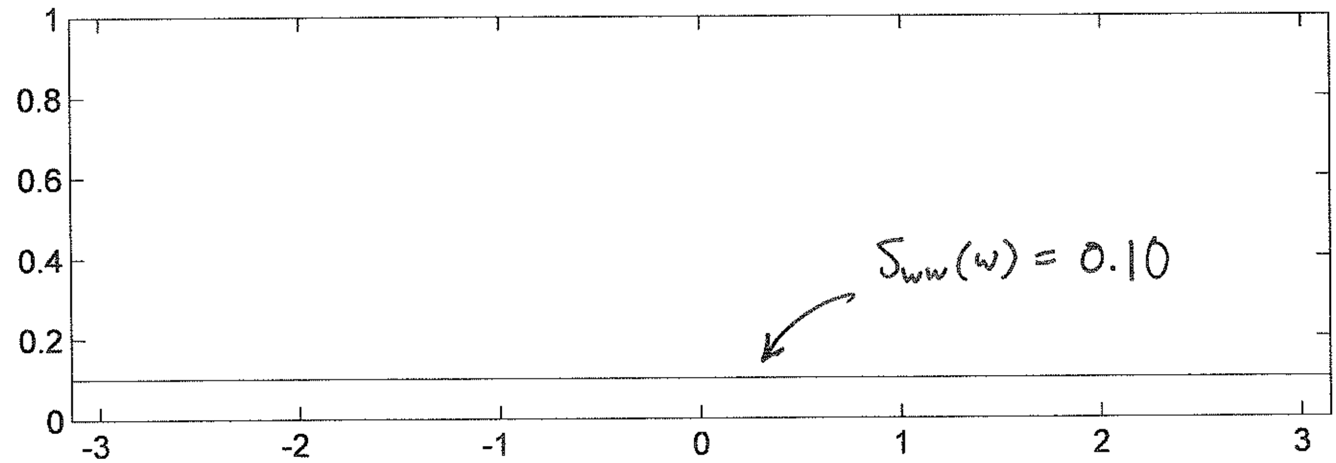
If the signal is strong (relative to noise) at frequency ω, then keep that frequency component...

Otherwise attenuate that component.

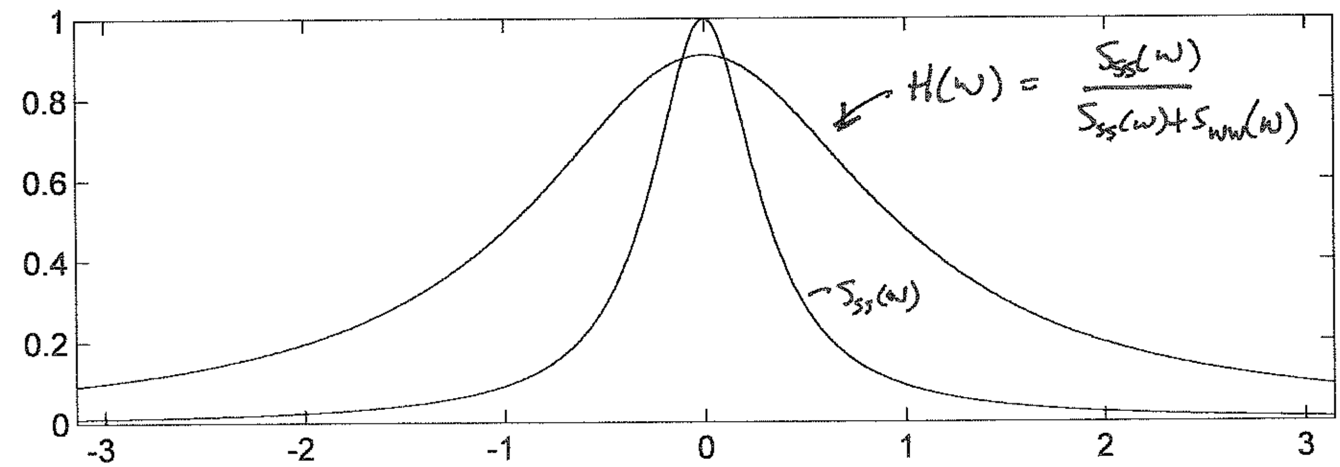
Ex.



1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
$S_{ss}(\omega) = \frac{1}{1+10\omega^2}$

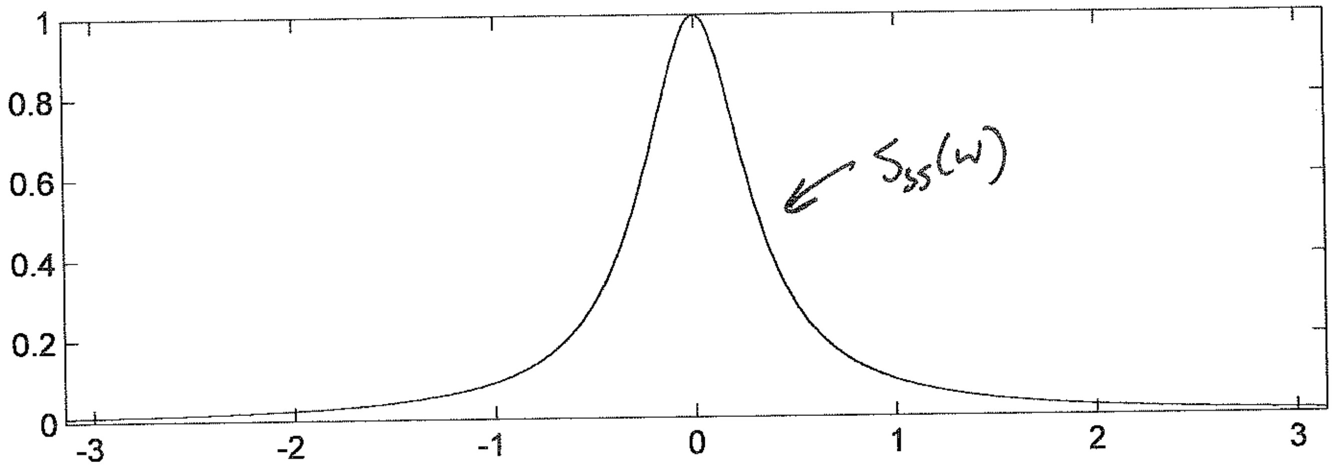


1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
$S_{ww}(\omega) = 0.10$

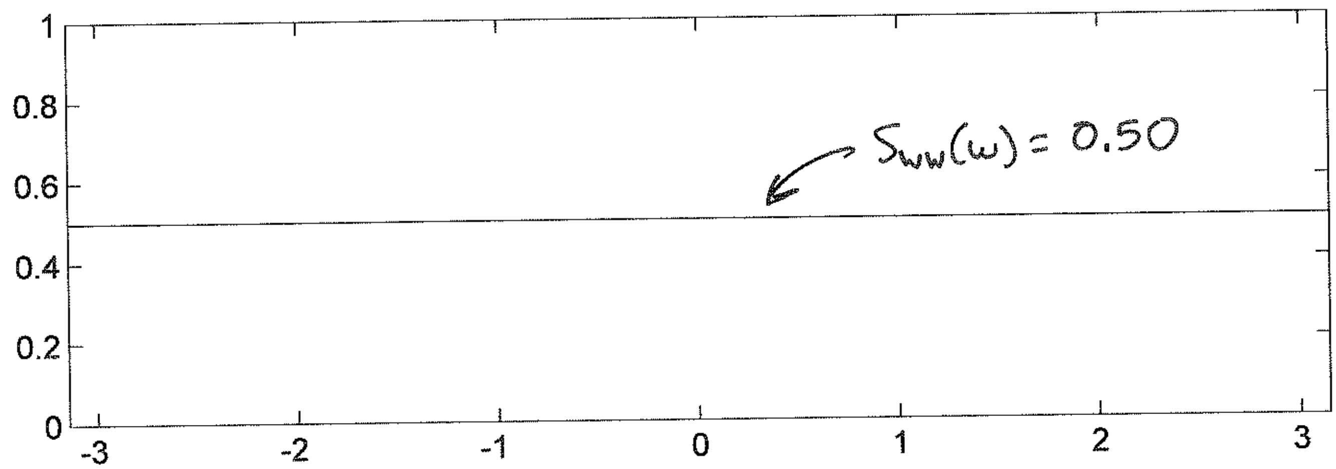


1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
$H(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega)+S_{ww}(\omega)}$
$S_{ss}(\omega)$

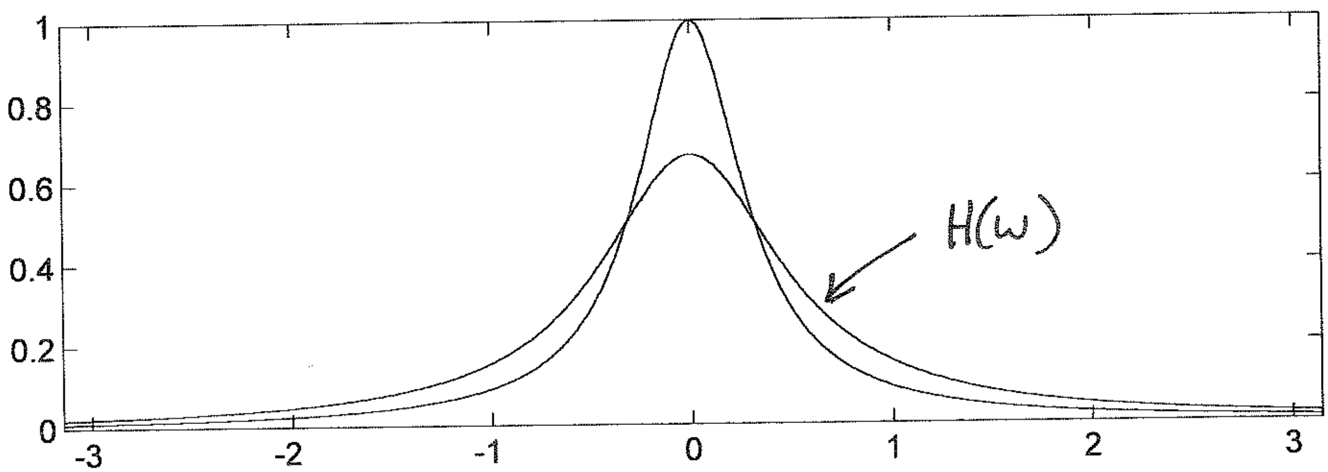
Ex.



$S_{ss}(\omega)$
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3

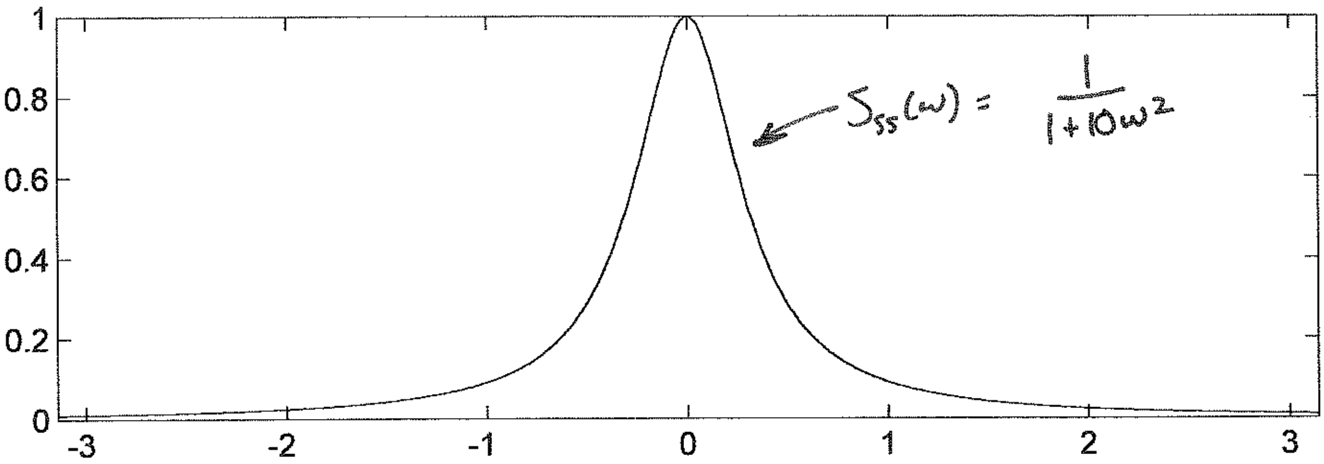


$S_{ww}(\omega) = 0.50$
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3

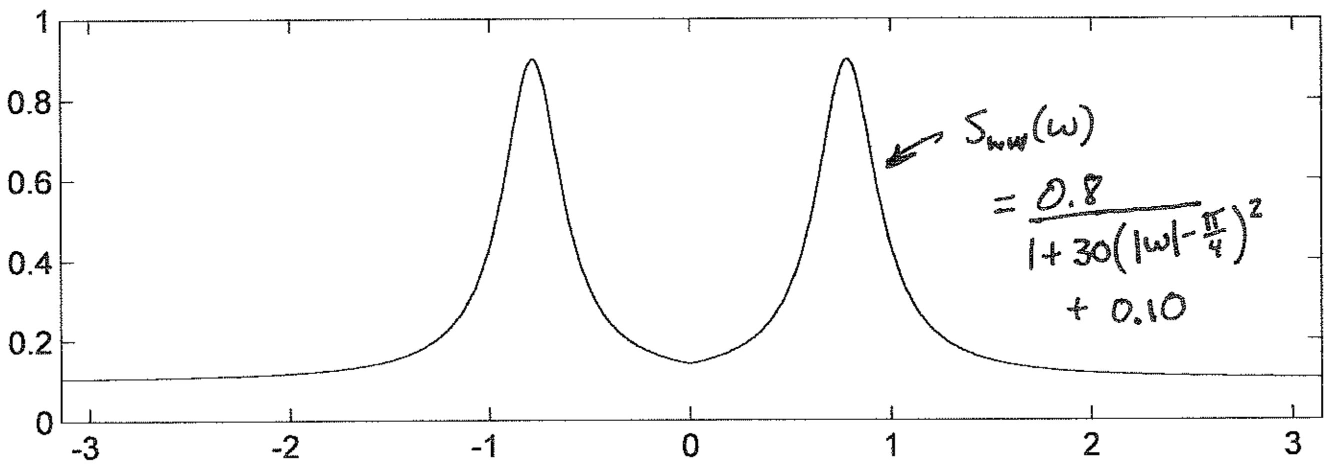


$H(\omega)$
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3

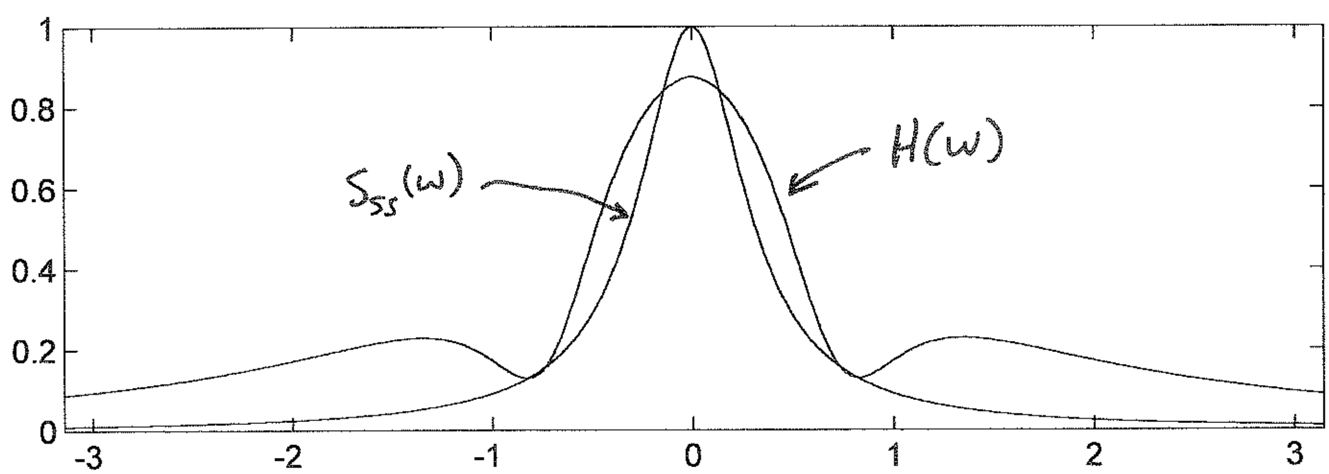
Ex.



$S_{ss}(\omega) = \frac{1}{1+10\omega^2}$

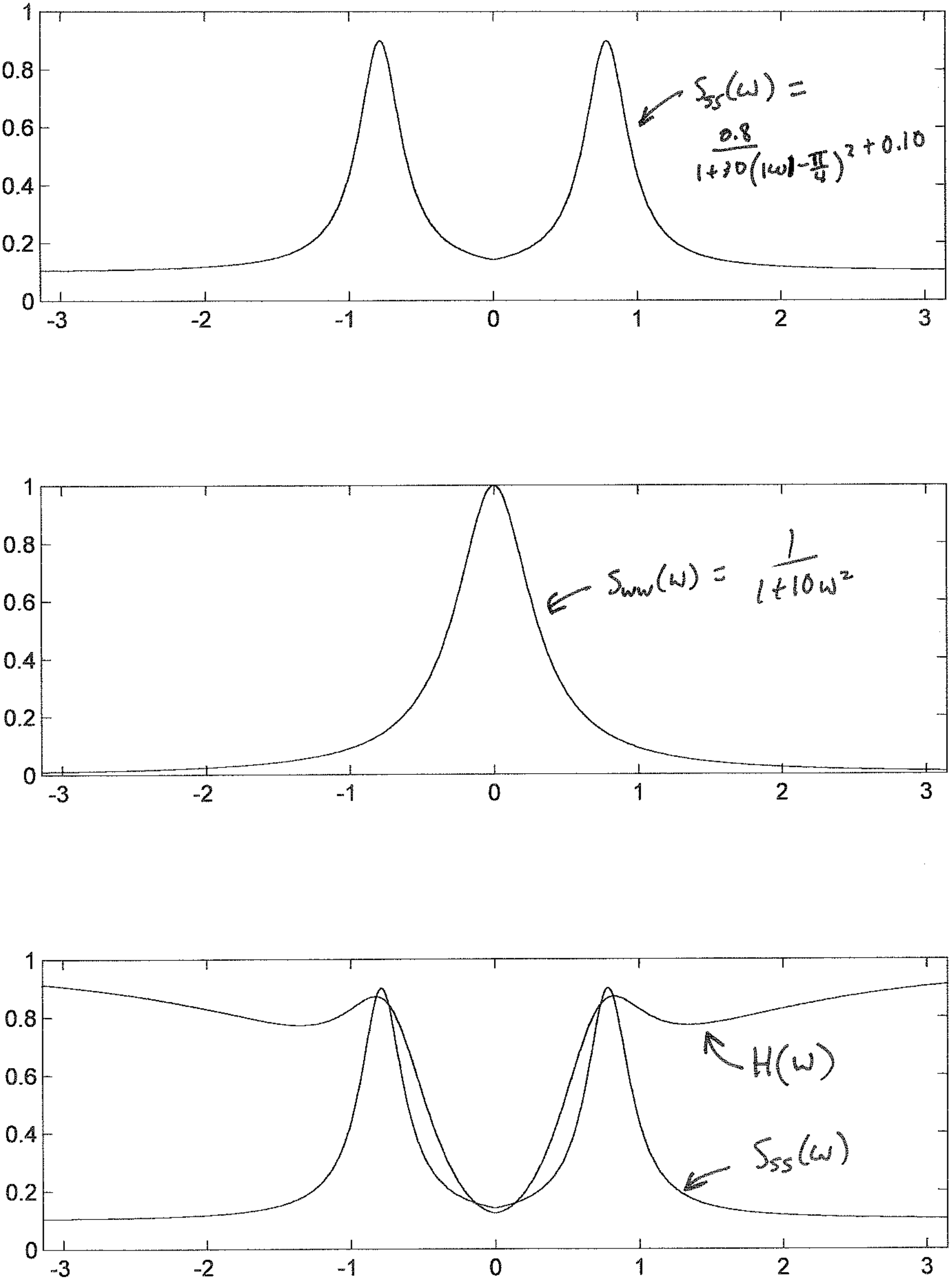


$S_{ww}(\omega) = \frac{0.8}{1+30(|\omega|-\frac{\pi}{4})^2} + 0.10$



$S_{ss}(\omega)$
$H(\omega)$

Ex. "One person's signal is another's noise"



FIR Wiener Filters

In general, the optimum Wiener filter is IIR. That is, its time support is unlimited. This may mean that the filter is not easily realizable. Instead, let's try to find the best FIR filter; a sort of constrained Wiener filter.

We again choose MSE as our optimality criterion, but force the filter $h[n]$ to be a length M FIR filter.

$$\hat{s}[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k]$$

$$MSE(h) = E\left[\left(s[n] - \sum_{k=0}^{M-1} h[k]\,x[n-k]\right)^2\right]$$

Again, the MSE is a quadratic function of $h[n]$ and the optimal filter satisfies the Wiener-Hopf equation:

$$\sum_{k=0}^{M-1} h[k]\, R_{xx}[m-k] = R_{xs}[m]$$

These equations can be written in matrix form as

$$\underset{(M\times M)}{\underline{R_{xx}}}\ \underset{(M\times 1)}{\underline{h}} = \underset{(M\times 1)}{\underline{R_{xs}}}$$

with

$$\underline{R}_{xx} = \begin{bmatrix} R_{xx}[0] & R_{xx}[1] & \cdots & R_{xx}[M-1] \\ R_{xx}[1] & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{xx}[1] \\ R_{xx}[M-1] & \cdots & R_{xx}[1] & R_{xx}[0] \end{bmatrix}$$

where I used the fact that
$R_{xx}[k] = R_{xx}[-k]$ for
stationary processes

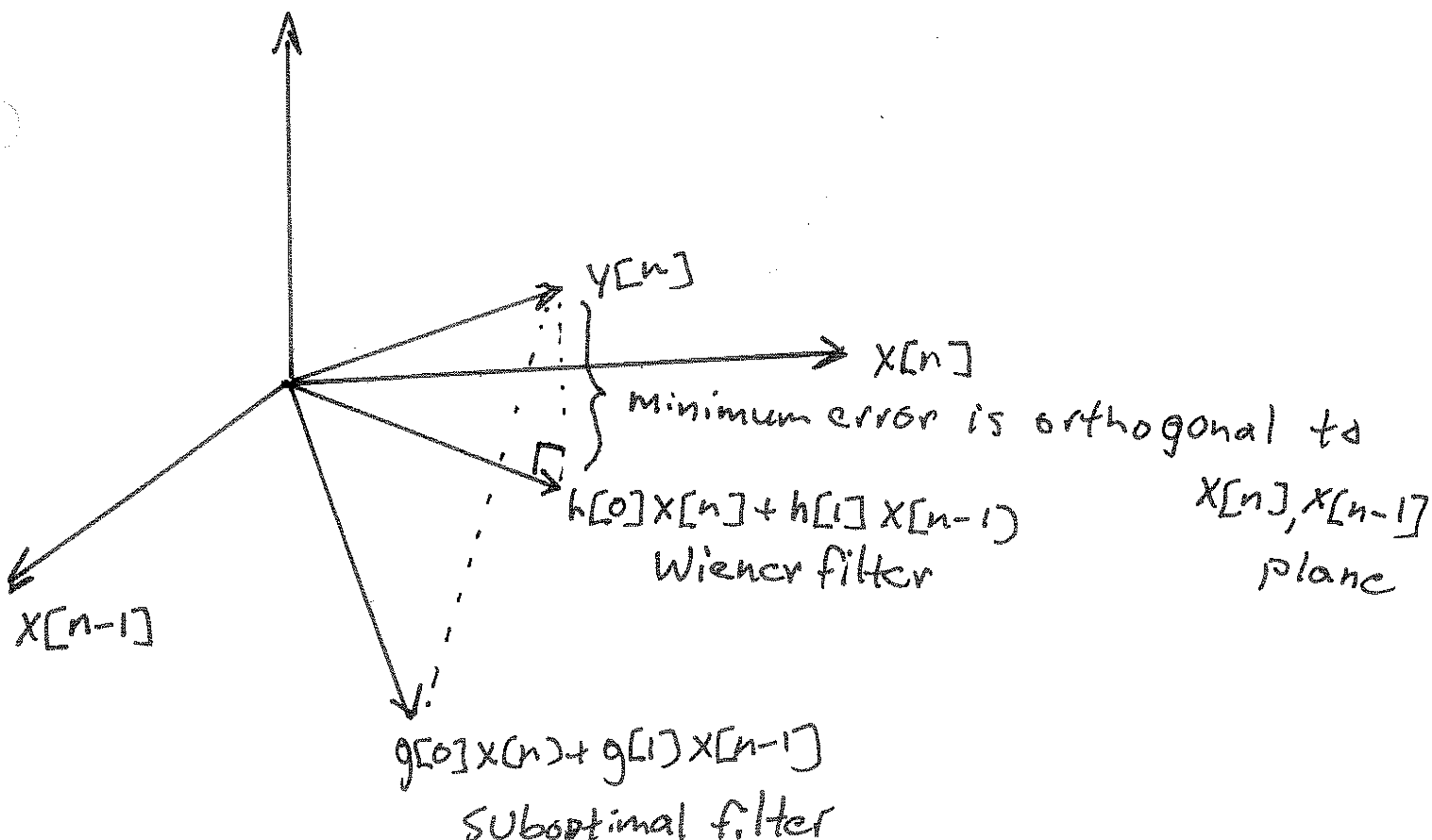
$$\underline{h} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[M-1] \end{bmatrix} \qquad \underline{R}_{xs} = \begin{bmatrix} R_{xs}[0] \\ R_{xs}[1] \\ \vdots \\ R_{xs}[M-1] \end{bmatrix}$$

Thus, the optimal Wiener filter is given by

$$\boxed{\underline{h} = \underline{R}_{xx}^{-1} \underline{R}_{xs}}$$

Orthogonality Revisited:

Suppose $M = 2$



To Come

- 2-d Wiener Filters for image processing
- Wiener deconvolution and image deblurring
- Wiener filtering and the 2-D FFT

Wiener Filtering via the FFT

We can use fast FFT-based convolution to speed-up the Wiener filtering process.

Recall: Wiener filter satisfies the Wiener-Hopf equation

$$\sum_{k} h[k] \Big(R_{ss}[m-k] + R_{ww}[m-k] \Big) = R_{ss}[m]$$

$$\Big\updownarrow \text{ DTFT}$$

$$H(\omega) \Big(S_{ss}(\omega) + S_{ww}(\omega) \Big) = S_{ss}(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{ww}(\omega)}$$

Suppose that we want to Wiener filter an N-point signal

$$x[n], \quad n = 0, \ldots, N-1.$$

To do this, in the DTFT domain we compute

DTFT of $\hat{s}[n]$ DTFT of $x[n]$

$$\hat{S}(\omega) = H(\omega) \cdot X(\omega)$$

Let's sample this equation in frequency at points $\omega = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$ to get

DFT of $h[n]$ DFT of $x[n]$ (N-point DFT)

$$\tilde{S}[k] = \underbrace{H[k] \cdot X[k]}_{\text{circular convolution in time}}$$

Taking inverse DFT we obtain

$$\tilde{s}[n] = h[n] \circledast x[n]$$

circular conv.

If $x[n]$ is properly zero-padded,

then $\tilde{s}[n] = \hat{s}[n]$ on $n = 0, \ldots, N-1$

true Wiener filter output via regular conv.

<u>Ex.</u> Suppose $x[n]$ and $h[n]$ are both N-point sequences.

Then zero-pad both to length $2N-1$ (or nearest power of two $\geq 2N-1$).

<u>Compute FFT of $x[n]$, $h[n]$:</u>

$O(N \log N)$ operations

<u>Multiply:</u>

$\tilde{S}[k] = H[k] \cdot X[k] \longrightarrow O(N)$ ops

<u>Inverse FFT:</u>

$\hat{s}[n]$, $O(N \log N)$ ops

Total: $O(N \log N)$

compare with direct conv. $O(N^2)$ ops

Signal Restoration Using the Wiener Filter

Suppose that we measure a distorted signal

$$s[n] \longrightarrow \underbrace{\boxed{g[n]}}_{\text{LTI}} \longrightarrow x[n]$$

$$x[n] = \sum_k g[k]\, s[n-k]$$

$$X(\omega) = G(\omega) S(\omega)$$

If $|G(\omega)| > 0$ for all $-\pi \le \omega \le \pi$, then we can recover $s[n]$ by computing

$$S(\omega) = \frac{X(\omega)}{G(\omega)}$$

then take inv DTFT of $S(\omega)$.

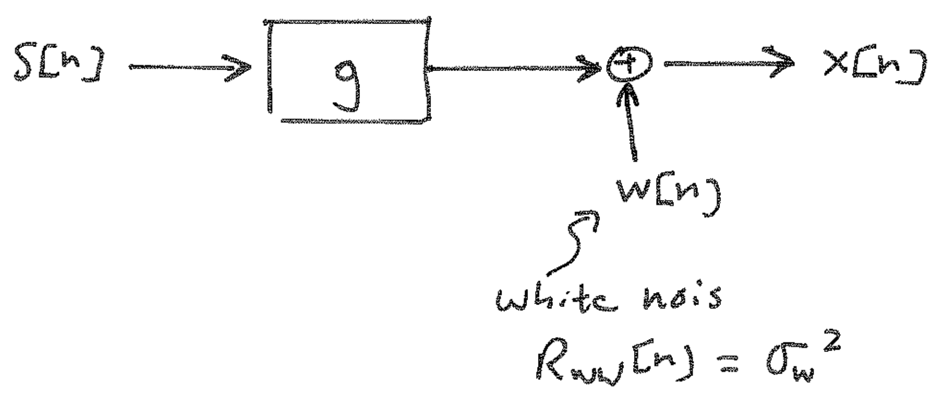
What if $G(\omega) = 0$ for some ω?

The filter $\frac{1}{G(\omega)} = G^{-1}(\omega)$ is called an <u>inverse filter</u>.

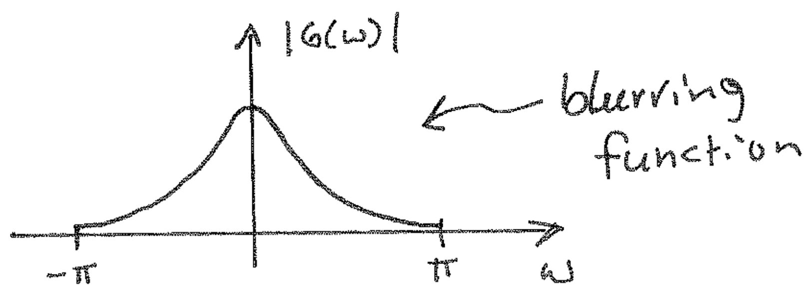
If $G(\omega) = 0$ for some ω, then $G^{-1}(\omega)$ does ot exist.

Even if $G^{-1}(\omega)$ exists, noise in our measurements may severely degrade results.

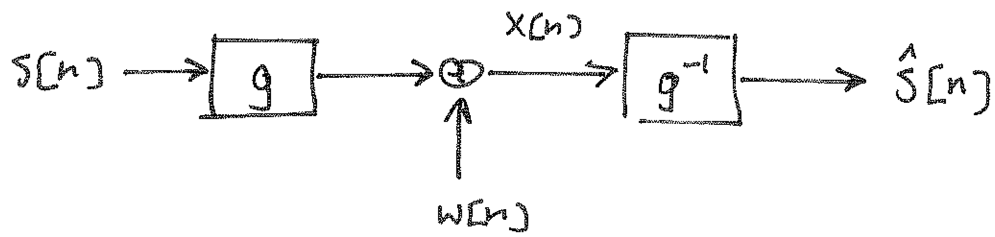
<u>Ex.</u>



Assume



Ex. (Cont.)



In frequency domain,

$$\hat{S}(\omega) = \frac{G(\omega)S(\omega)}{G(\omega)} + \frac{W(\omega)}{G(\omega)}$$

$$= S(\omega) + \frac{W(\omega)}{G(\omega)}$$

IDTFT

$$\Rightarrow \quad \hat{s}[n] = s[n] + \underbrace{g^{-1}[n] * w[n]}_{\text{noise at output of filter, } w'[n]}$$

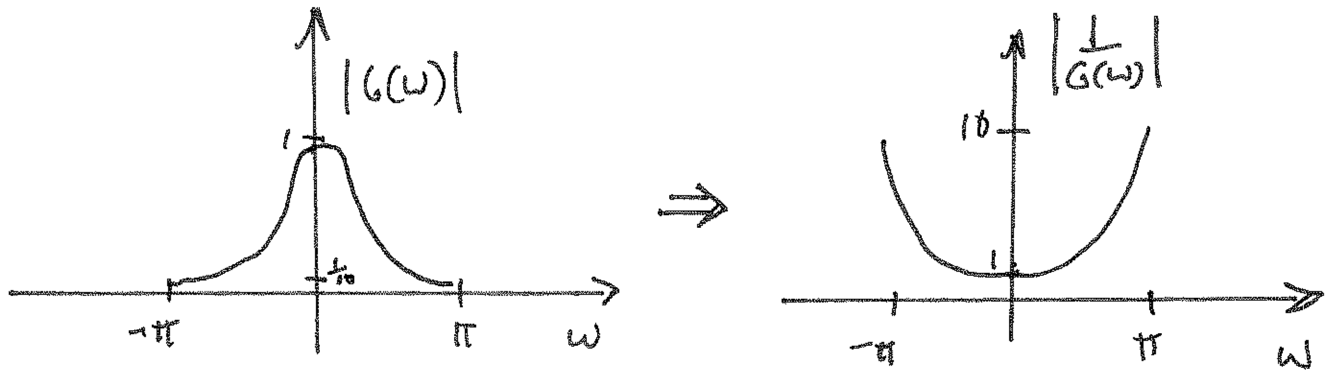
where $g^{-1}[n] = \text{IDTFT}\left(\frac{1}{G(\omega)}\right)$

Power spectral density of output noise:

$$S_{w'w'}(\omega) = \frac{S_{ww}(\omega)}{|G(\omega)|^2} = \frac{\sigma_w^2}{|G(\omega)|^2}$$

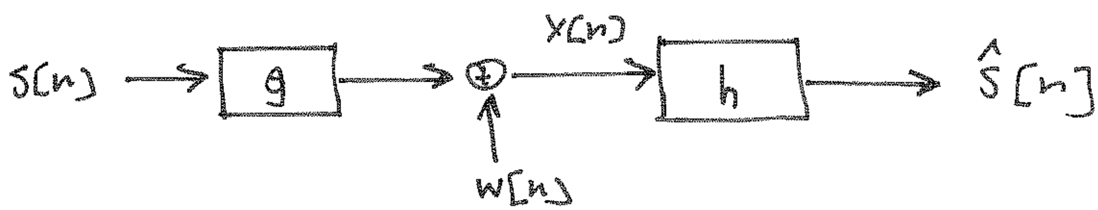
const. for all ω

Recall



The inverse filter drastically amplifies high frequency noise!!

Wiener Filter Solution:



Design $h[n]$ to minimize the MSE $E[(s[n]-\hat{s}[n])^2]$

As always, the optimal MMSE Wiener filter satisfies

$$R_{xs}[m] = \sum_k h[k]\, R_{xx}[m-k]$$

or

$$S_{xs}(\omega) = H(\omega)\, S_{xx}(\omega)$$

$R_{XS}[m] =$

$$E\Big[\, X[n-m]\, s[n] \,\Big] = E\Big[\Big(\sum_k g[k]\, s[n-m-k] + w[n-m]\Big)\cdot [n]\Big]$$

$$= E\Big[\sum_k g[k]\, s[n-m-k]\, s[n]\Big] ,$$

since $w[n]$ and $s[n]$ are indep. & zero-mean

$$R_{XS}[m] = \sum_k g[k]\, R_{SS}[m+k]$$

note $+k$ not $-k$
$\Rightarrow$ correlation not conv.

$\updownarrow$ DTFT

$$\boxed{S_{XS}(\omega) = G^{*}(\omega)\, S_{SS}(\omega)}$$

$$R_{XX}[m] = E\big[\, x[n]\, x[n-m] \,\big]$$

$$= E\Big[\Big(\sum_k g[k]\ [n-k] + w[n]\Big)\Big(\sum_\ell g[\ell]\, s[n-m-\ell] + w[n-m]\Big)\Big]$$

$$= E\Big[\sum_k \sum_\ell g[k]\, g[\ell]\, s[n-k]\, s[n-m-\ell]\Big] + E\Big[\, w[n]\, w[n-m] \,\Big]$$

by indep. & zero-mean assumptions

$$= \sum_k \sum_\ell g[k]\, g[\ell]\, R_{SS}[m+\ell-k] + R_{WW}[m]$$

$$R_{xx}[m] = \sum_{k} \sum_{l} g[k]\, g[l]\, R_{ss}[m+l-k] + R_{ww}[m]$$

$$\Big\updownarrow \text{ DTFT}$$

$$S_{xx}(\omega) = G^*(\omega)\, G(\omega)\, S_{ss}(\omega) + S_{ww}(\omega)$$

$$\boxed{S_{xx}(\omega) = |G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega)}$$

Thus,

$$S_{xs}(\omega) = H(\omega)\, S_{sx}(\omega)$$

$$\Rightarrow$$

$$G^*(\omega)\, S_{ss}(\omega) = H(\omega)\left(|G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega)\right)$$

$$\Rightarrow \boxed{H(\omega) = \frac{G^*(\omega)\, S_{ss}(\omega)}{|G(\omega)|^2 S_{ss}(\omega) + S_{ww}(\omega)}}$$

Optimal Wiener "restoration" filter

Comparison with simple inv. filter:

If $S_{ss}(\omega) >> S_{ww}(\omega)$, then

$$H(\omega) \approx \frac{1}{G(\omega)}$$

If $S_{ss}(\omega) << S_{ww}(\omega)$, then

$$H(\omega) \approx 0.$$

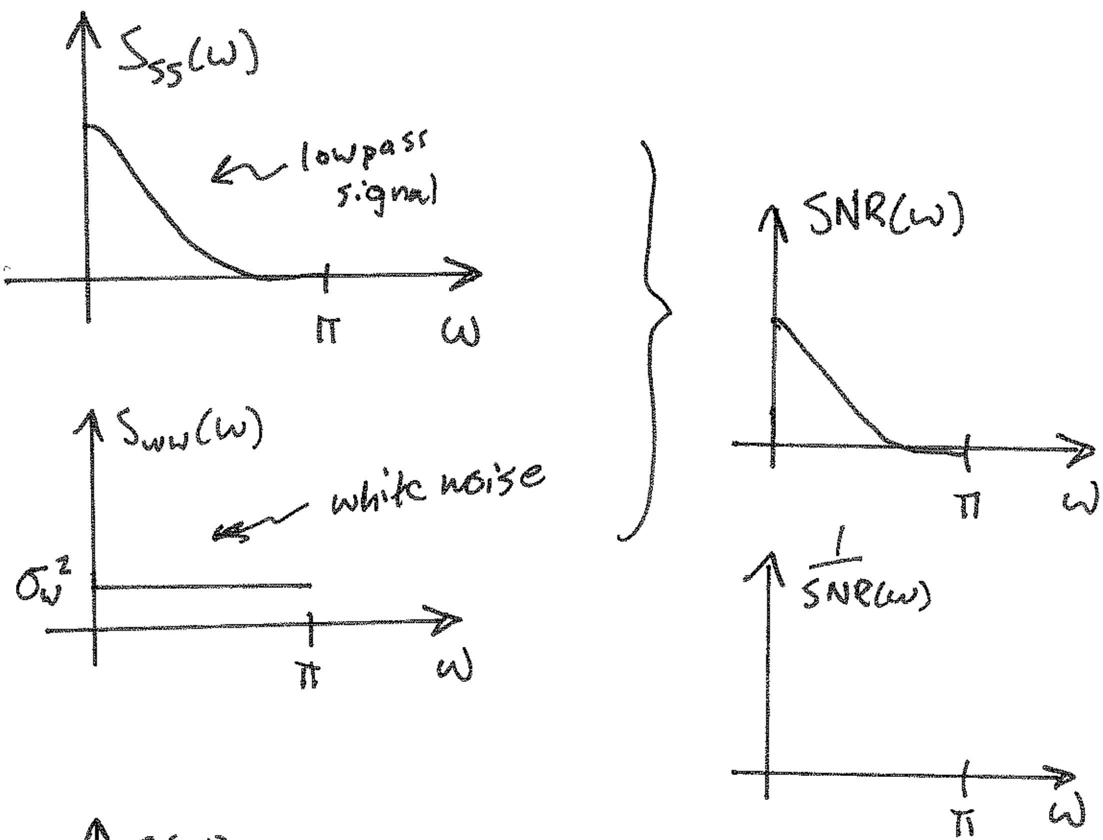
Note:

$$H(\omega) = \frac{G^*(\omega)}{|G(\omega)|^2 + \frac{1}{SNR(\omega)}}$$

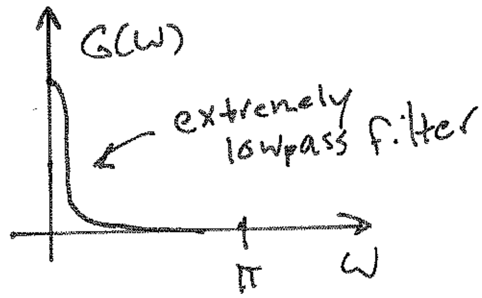
where $SNR(\omega) \equiv \dfrac{S_{ss}(\omega)}{S_{ww}(\omega)}$

* as long as $SNR(\omega) < \infty$ (we have some noise)
$H(\omega)$ is well-defined!!

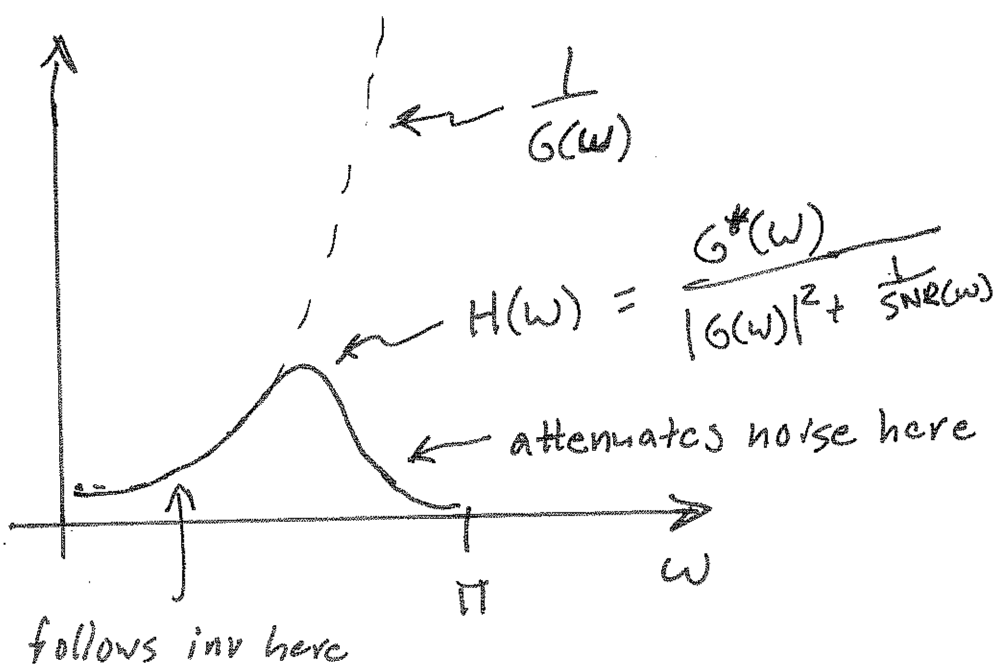
Ex.



$S_{SS}(\omega)$
lowpass signal
π
ω
$S_{WW}(\omega)$
white noise
σ_w^2
π
ω
$SNR(\omega)$
π
ω
$\frac{1}{SNR(\omega)}$
π
ω

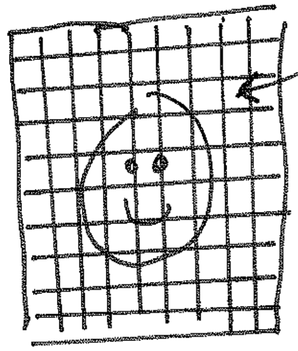


$G(\omega)$
extremely lowpass filter
π
ω



$\frac{1}{G(\omega)}$
$H(\omega) = \frac{G^*(\omega)}{|G(\omega)|^2 + \frac{1}{SNR(\omega)}}$
attenuates noise here
follows inv here
π
ω

Application to Image Restoration



row index

column index

$x[n_1, n_2]$

(n_1, n_2) pixel of image

image samples $\equiv$ pixels

2-d Convolution

$$x[n_1,n_2] * g[n_1,n_2]$$

$$= \sum_{k_1} \sum_{k_2} g[k_1,k_2]\, x[n_1-k_1, n_2-k_2]$$

Ex.

$$g[n_1,n_2] = \begin{cases} \frac{1}{4}, & n_1, n_2 = 0,1 \\ 0, & \text{otherwise} \end{cases}$$

$\Longrightarrow$ $x[n_1,n_2] * g[n_1,n_2]$ results in 4-pixel averaging ("smoothing") of original image $x[n_1,n_2]$.

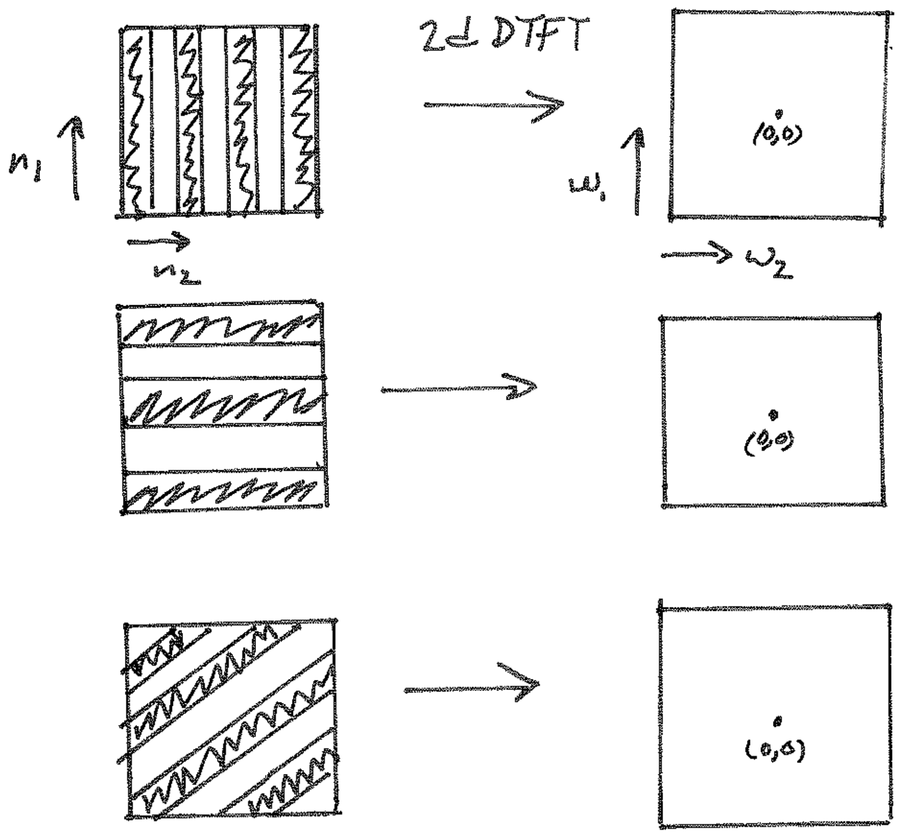
2-d DTFT

freq in row direction

$$X(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-j(\omega_1 n_1 + \omega_2 n_2)}$$

freq in column direction

Ex.



$$g[n_1,n_2] * x[n_1,n_2] \overset{\text{DTFT}}{\longleftrightarrow} G(\omega_1,\omega_2)\, X(\omega_1,\omega_2)$$

2-d DFT $\quad x[n_1,n_2]$ $N \times N$ image

$$X[k_1,k_2] = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x[n_1,n_2]\, e^{-j\left(\frac{2\pi k_1 n_1}{N} + \frac{2\pi k_2 n_2}{N}\right)}$$

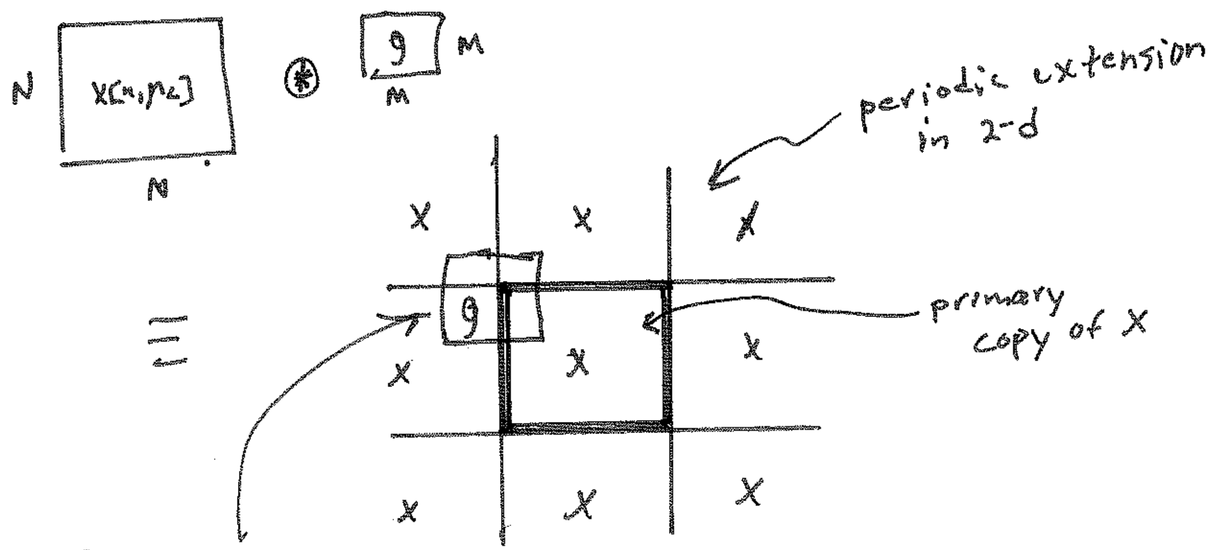
$$k_1, k_2 = 0, \ldots, N-1$$

$$x[n_1,n_2] = \frac{1}{N^2}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} X[k_1,k_2]\, e^{j\frac{2\pi k_1 n_1}{N}}\, e^{j\frac{2\pi k_2 n_2}{N}}$$

$$G[k_1,k_2] \cdot X[k_1,k_2] \overset{\text{DFT}}{\longleftrightarrow} g[n_1,n_2] \circledast x[n_1,n_2]$$

2-d circular convolution

2-d Circular Conv:



reg. convolve
g with periodically extended version of x

2-d FFT:

$x[n_1, n_2]$ is $N \times N$ image

Note: DFT is separable in rows and columns

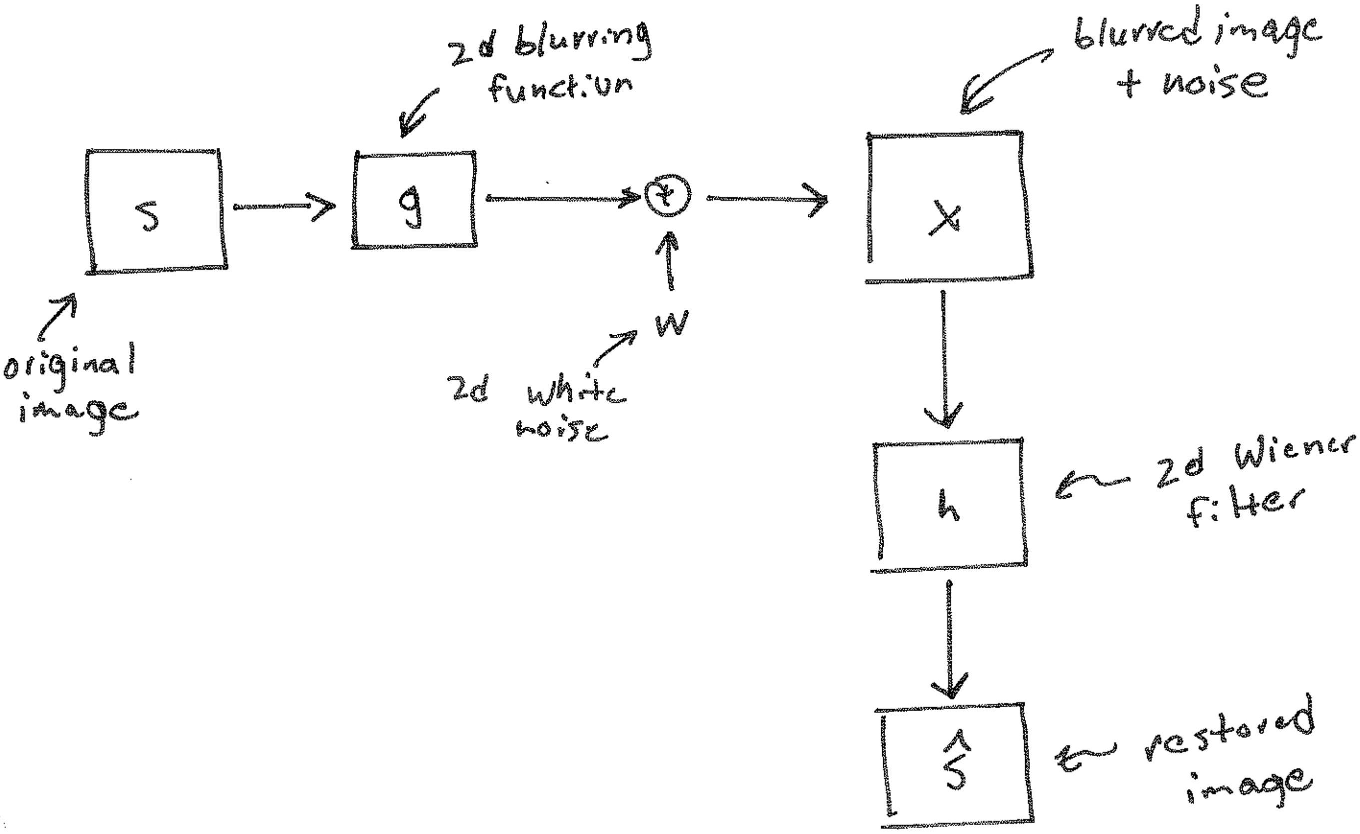
$$X[k_1, k_2] = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} x[n_1, n_2]\, e^{-j\frac{2\pi k_1 n_1}{N}}\, e^{-j\frac{2\pi k_2 n_2}{N}}$$

$$= \sum_{n_1=0}^{N-1} \underbrace{e^{-j\frac{2\pi n_1 k_1}{N}} \underbrace{\sum_{n_2=0}^{N-1} x(n_1, n_2)\, e^{-j\frac{2\pi k_2 n_2}{N}}}_{\text{1-d FFTs of each column}}}_{\text{then 1-d FFT along each row}}$$

Thus, using 1-d FFTs, the 2-d DFT can be computed in

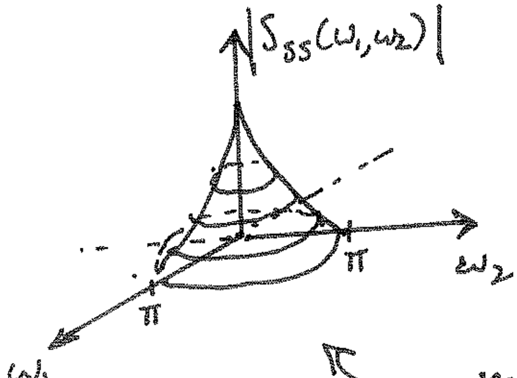
$$O(N^2 \log N^2) \text{ operations}$$

Image Deblurring:



Model image as 2-d random process:

$$S_{ss}(\omega_1, \omega_2) = \text{radially symmetric 2-d function}$$



mostly low freq energy (images mostly smooth)
but some high freqs, too (edges)

2-d Wiener Filter

$$H(\omega_1,\omega_2) = \frac{G^*(\omega_1,\omega_2)\, S_{ss}(\omega_1,\omega_2)}{|G(\omega_1,\omega_2)|^2 S_{ss}(\omega_1,\omega_2) + S_{ww}(\omega_1,\omega_2)}$$

Sample

$$\Rightarrow \quad H[k_1,k_2] = H\left(\frac{2\pi k_1}{N}, \frac{2\pi k_2}{N}\right)$$

$k_1, k_2 = 0, \ldots, N-1$

DFT Implementation:

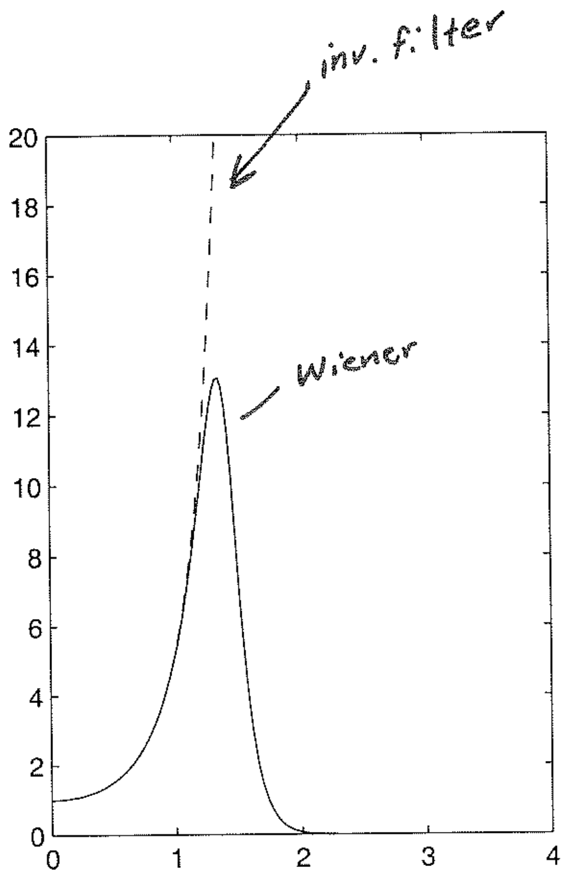
$$\tilde{S}[k_1,k_2] = \underbrace{H[k_1,k_2]\cdot X[k_1,k_2]}_{\text{equivalent to 2-d circular conv. in space}}$$

fast!!
$O(N^2 \log N^2)$

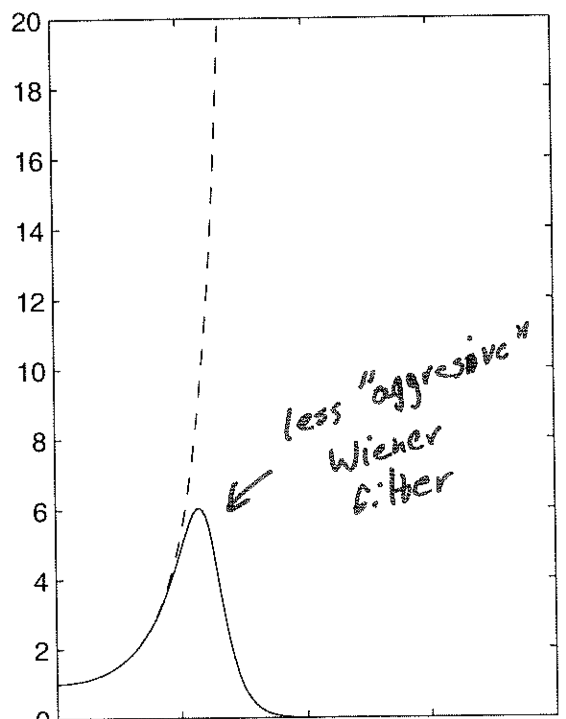
some possible distortion esp. near boundary of image

Ex.

Blurring with little noise



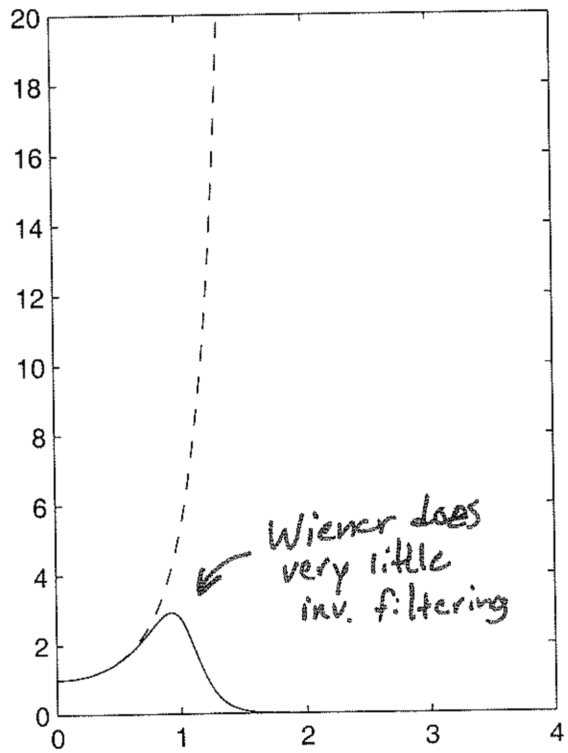
Blurring with moderate noise



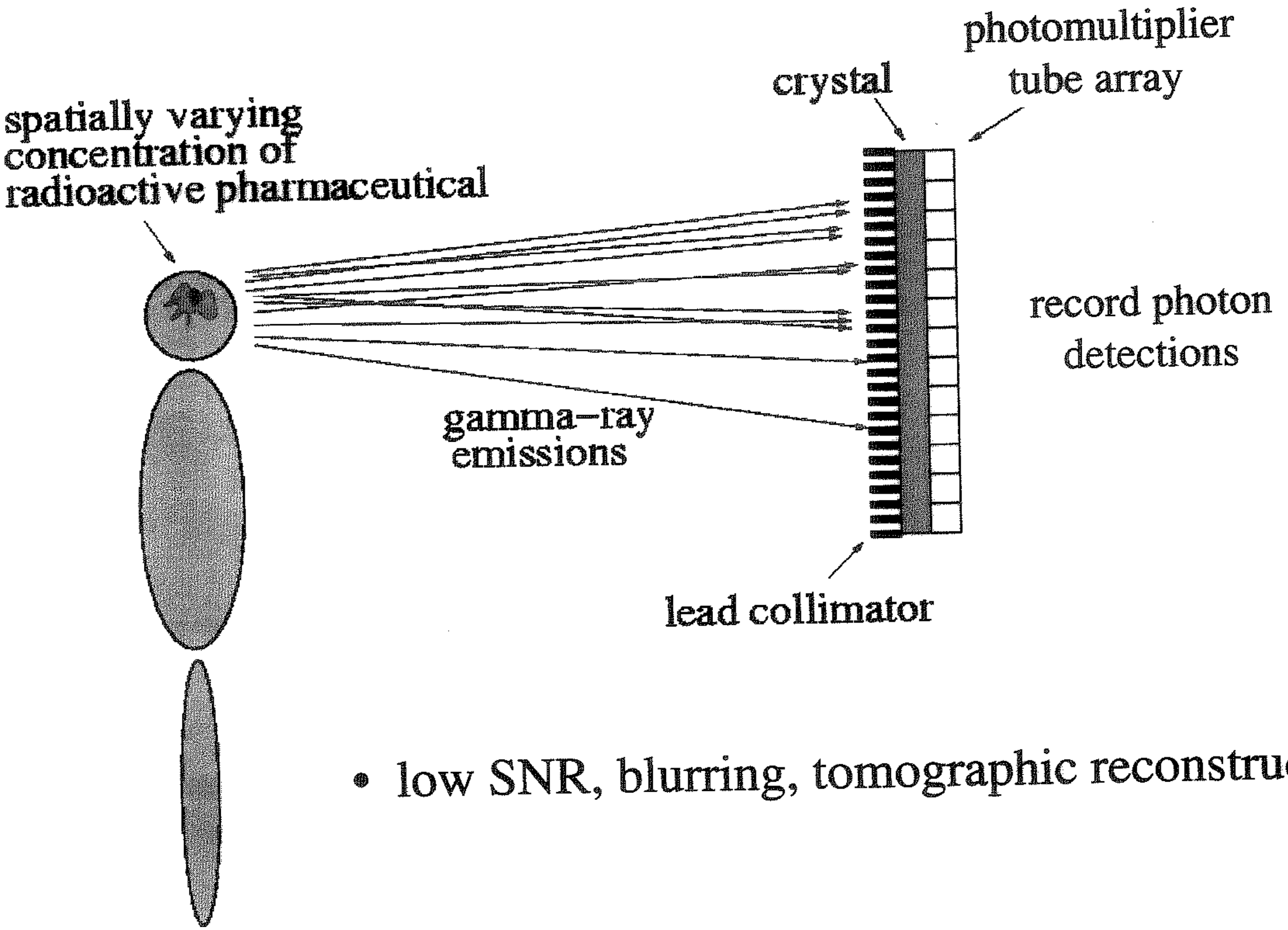
noise "artifact"

Blur with heavy noise

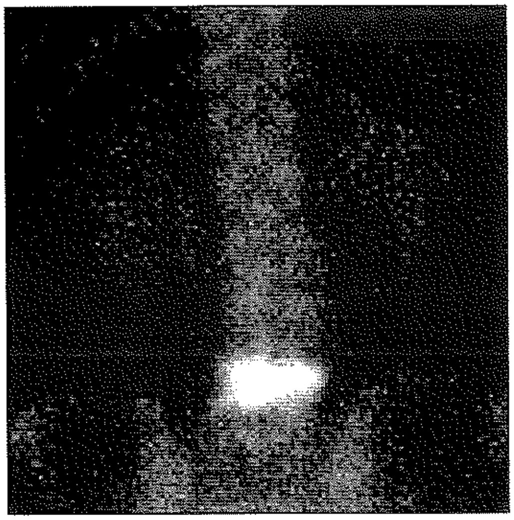
severe noise artifacts



Nuclear Medicine Imaging

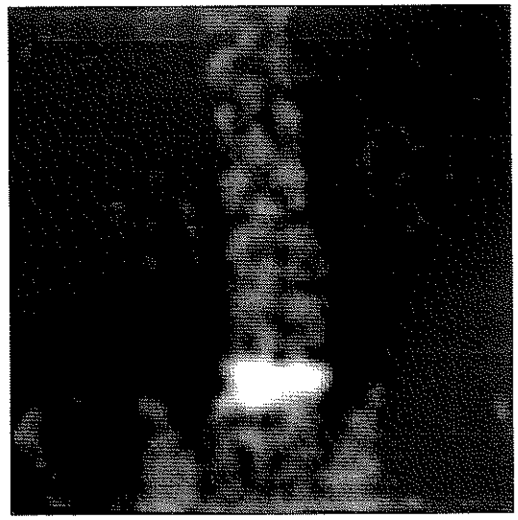


- low SNR, blurring, tomographic reconstruction raw image



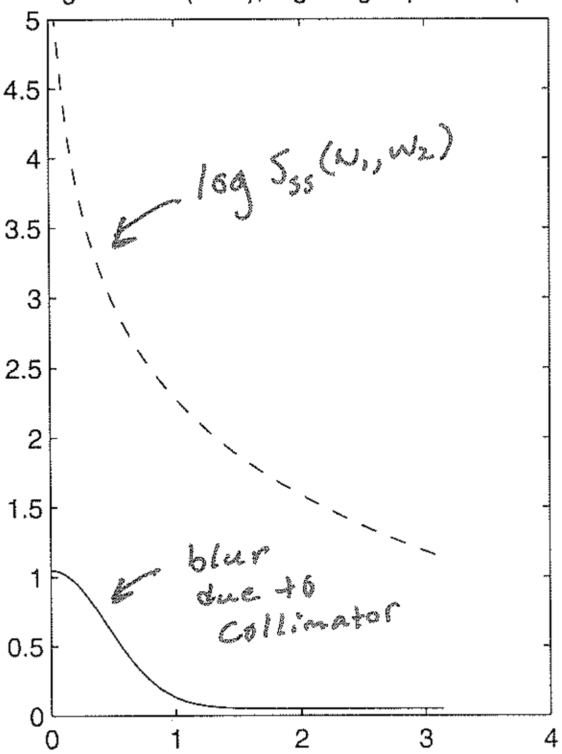
Nuclear Medicine
Spine image

Wiener filter restoration



Wiener filter
restoration

blurring function (solid), log image spectrum (dashed)



Wiener filter (solid), inverse filter (dashed)

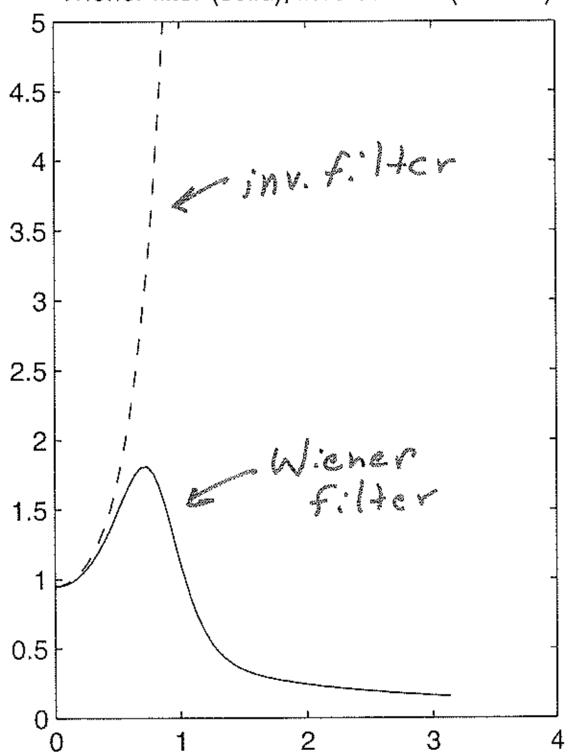


Image Power Spectra

What is a good model for

$S_{ss}(\omega_1, \omega_2)$?

Reasonable / Desirable Properties :

Images consist of "smooth" regions and patches seperated by boundaries / edges

"smooth" $\Rightarrow$ low frequency

edges $\Rightarrow$ high frequency

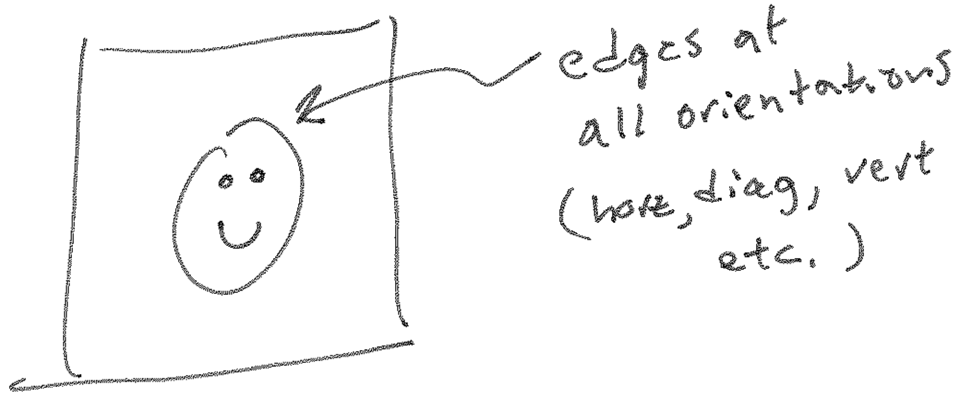
How many pixels are "edge pixels" and how many are "smooth patch pixels"?

Images are mostly "smooth patch pixels" (e.g., 90% of pixels). The rest are edge/texture pixels (e.g., 10%)

$\Rightarrow$ images have more energy at low freqs than at high freqs

Image edges, boundaries, textures occur at all possible orientations.

<u>Ex.</u>



$\Rightarrow$ freq characteristics (energy) should be fairly symmetric in the radial (angular) direction

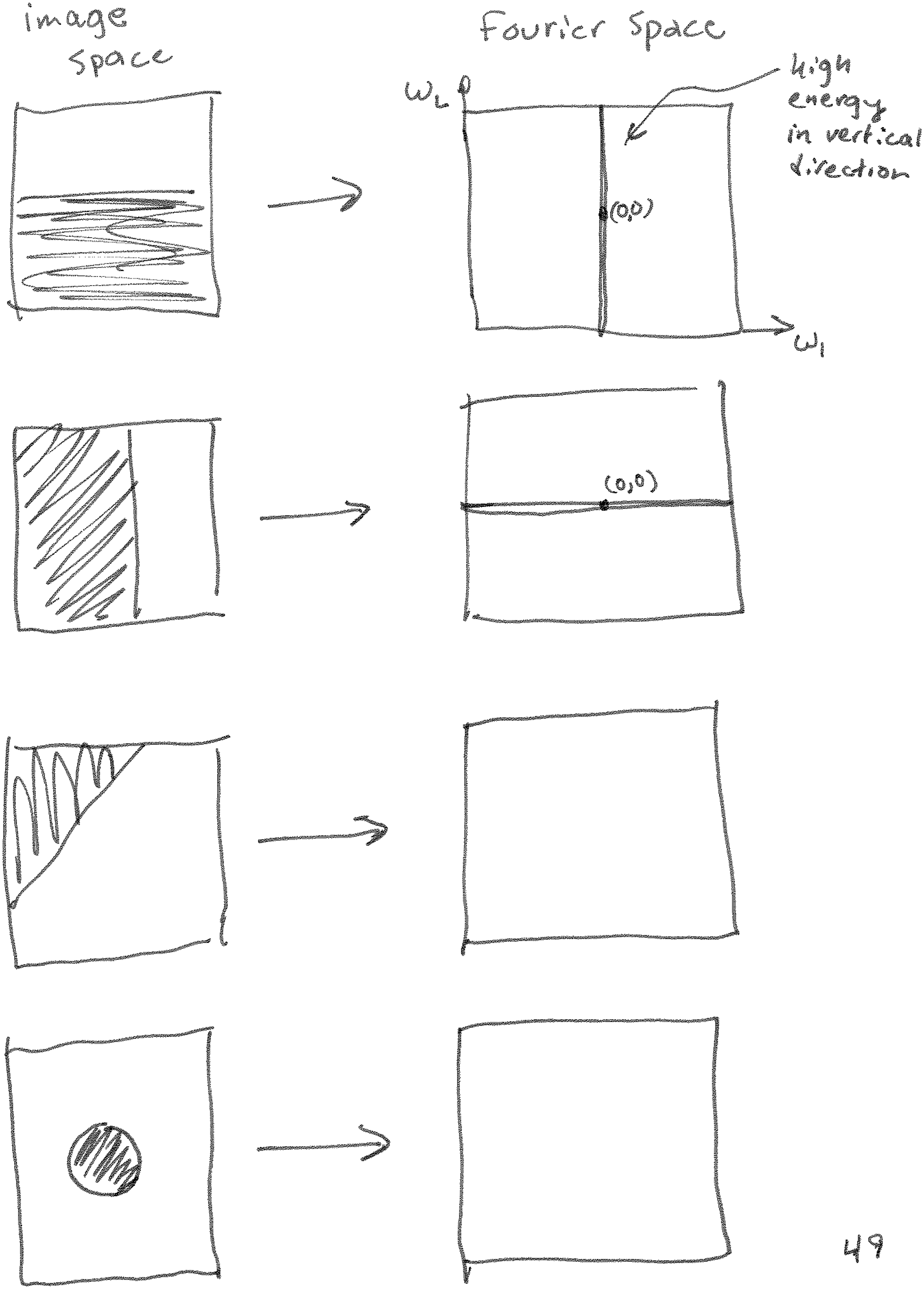
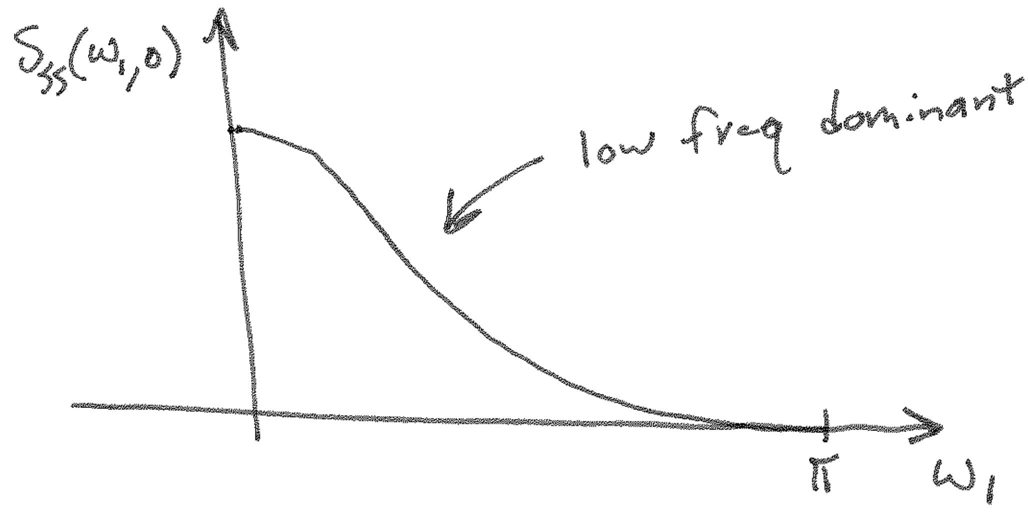


image space
Fourier space
ω_2
high energy in vertical direction
(0,0)
ω_1
(0,0)

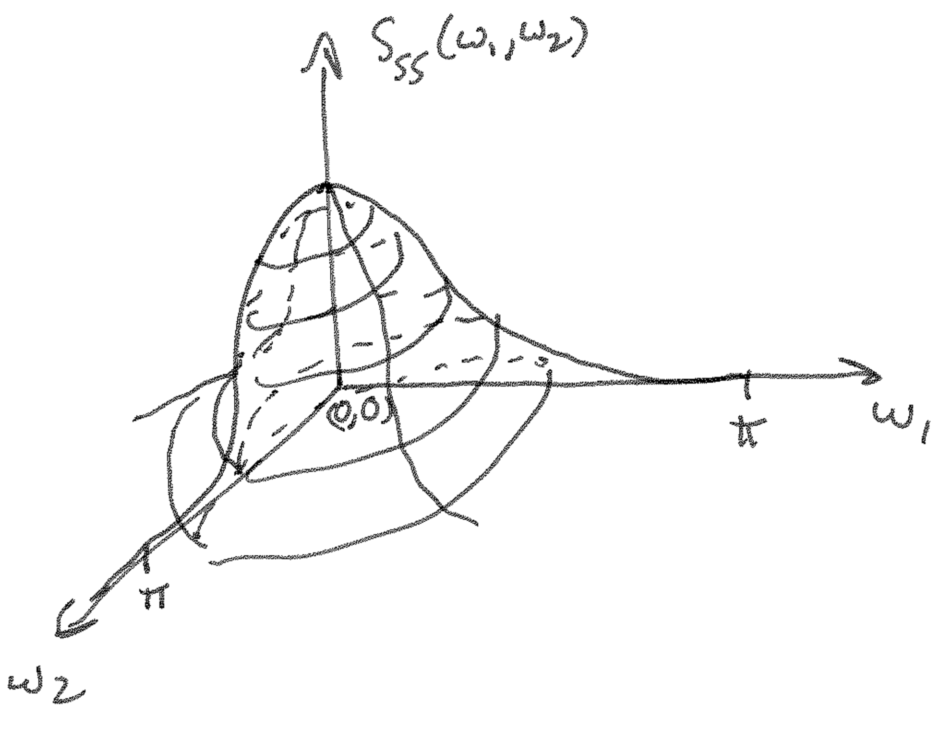
Basic model for $S_{ss}(\omega_1, \omega_2)$:

(i)



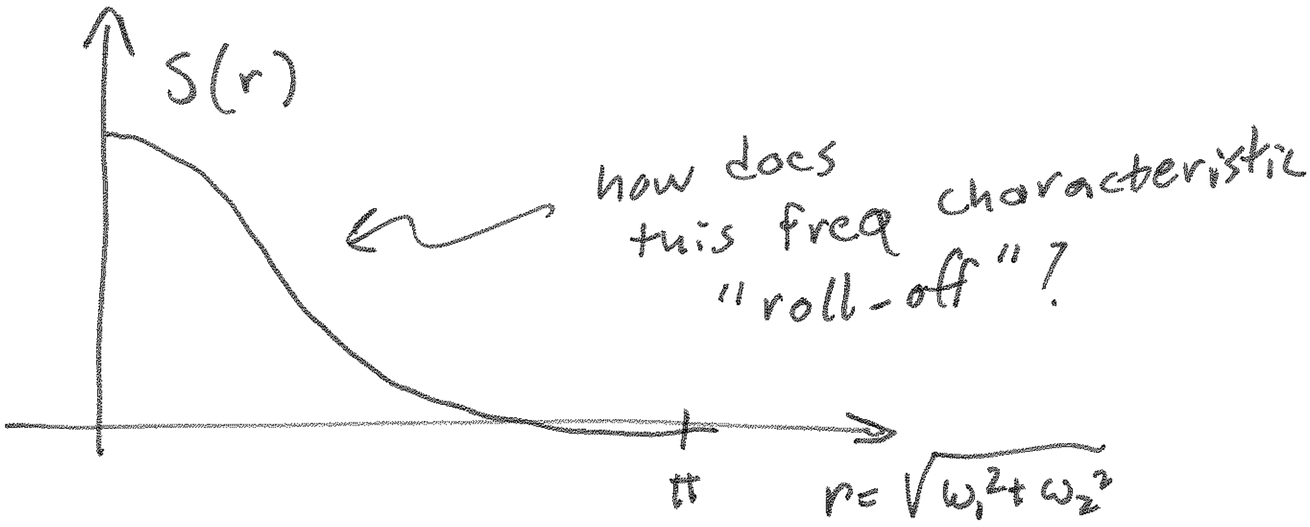
(ii) $S_{ss}(\omega_1, \omega_2)$ depends only on

$$r = \sqrt{\omega_1^2 + \omega_2^2} \quad \text{(radial symmetry)}$$



Modelling the radial "profile"

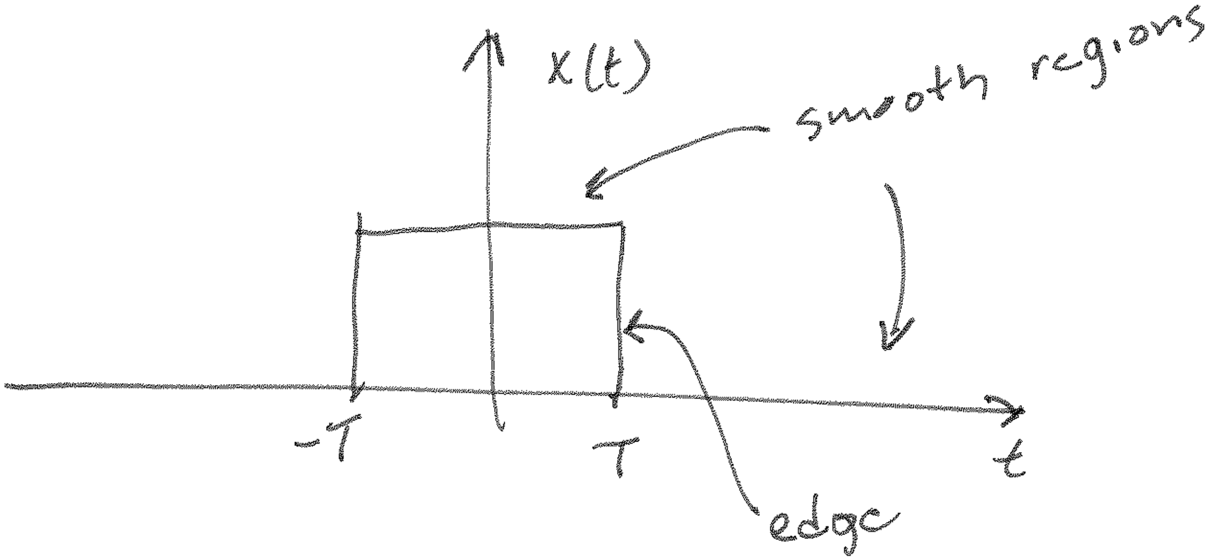
$$S_{ss}(\omega_1, \omega_2) \equiv S_{ss}(r, \phi) = S(r)$$



Think about simple 1-D function with the basic image "structure"

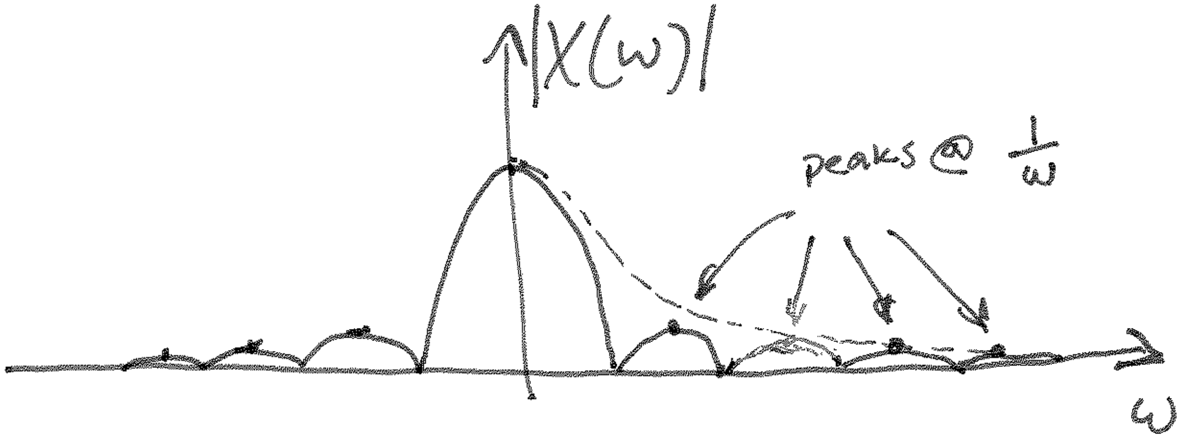
i.e., smooth regions separated by edges

A 1-D "image":



CTFT:

$$X(\omega) = \frac{\sin \omega T}{\omega} \quad \Leftarrow \text{sinc}$$



$$|X(\omega)| \le \frac{1}{|\omega|}$$

The CTFT decays like

$$|X(\omega)| \sim \frac{1}{|\omega|}$$

so the energy / power decays like

$$|X(\omega)|^2 \quad \frac{1}{|\omega|^2}$$

1-D "images" are random superpositions of blocks like this producing a power spectrum

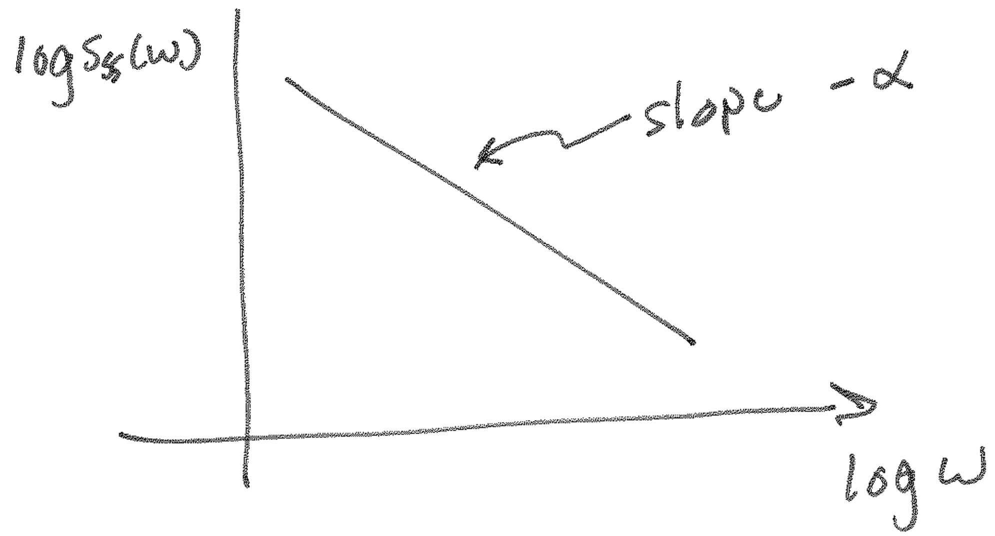
$$S_{ss}(\omega) \propto \frac{1}{|\omega|^2}$$

This called a "1 over frequency" or $1/f$ power spectrum.

In general, a $1/f$ spectrum has the form

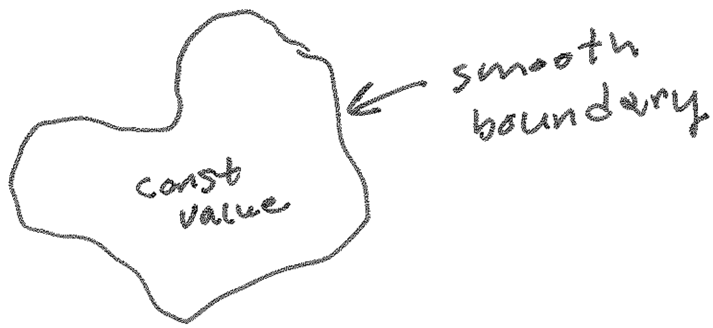
$$S_{ss}(\omega) = \frac{\text{Const.}}{|\omega|^\alpha}, \quad \alpha > 0.$$

Notice



So given a power spectrum it is easy to determine the correct α.

We can think of images (2-D) as being random superpositions of 2-D shapes with smooth boundaries and smooth surfaces



This leads to symmetric (radially) 2-D power spectra with $1/f$ characteristics. For real, natural images (collected by cameras) experiments have shown that images behave like

$$S_{ss}(r) \propto \frac{1}{|r|^{\alpha}}$$

$$1 \le \alpha \le 3$$

In summary, a reasonable model for image power spectra if

$$S_{ss}(\omega_1, \omega_2) = \frac{\sigma^2}{\left(\sqrt{\omega_1^2 + \omega_2^2 + 1}\right)^\alpha}$$

Where σ^2 is the DC power and α is the exponent of the $1/f$ decay.

With this model, we can design a 3-parameter Wiener restoration filter depending on α, σ^2, and σ_v^2 (the white noise power).