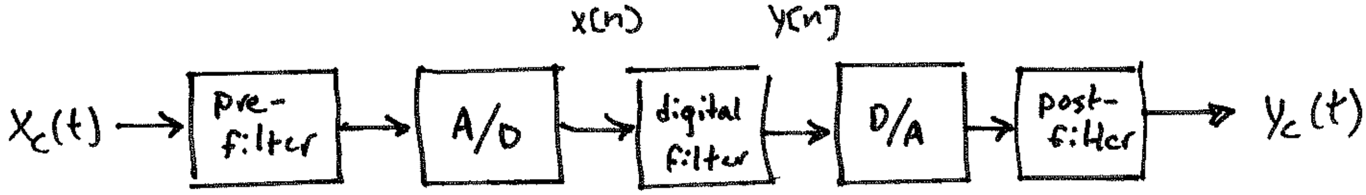
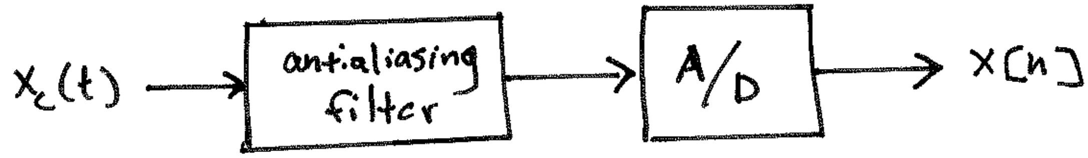
Real-World DSP systems



Practical Issues

1. $x_c(t)$ bandlimited?
2. ideal lowpass filters?
3. ideal impulse trains?
4. quantization

Antialiasing Prefilter

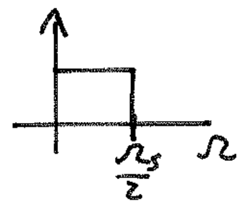


continuous-time filter

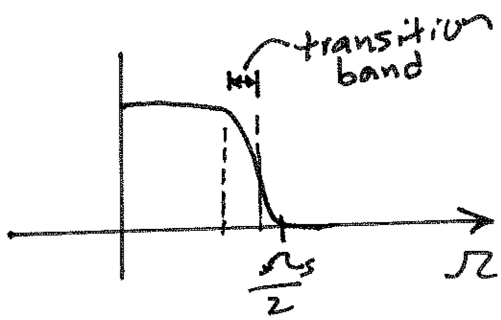
Twofold Purpose:

1. bandlimits input signal $x_c(t)$ to desired range (i.e., $|\Omega| \le \Omega_s/2$)
2. limits additive noise and other interference that may corrupt desired signal

Ideally: perfect lowpass filter

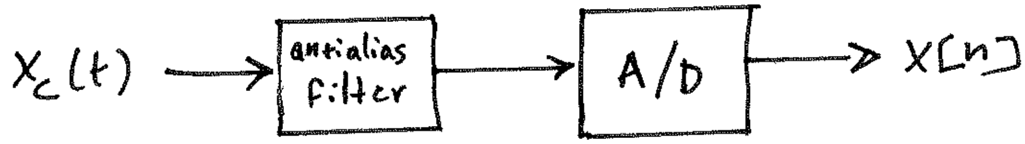


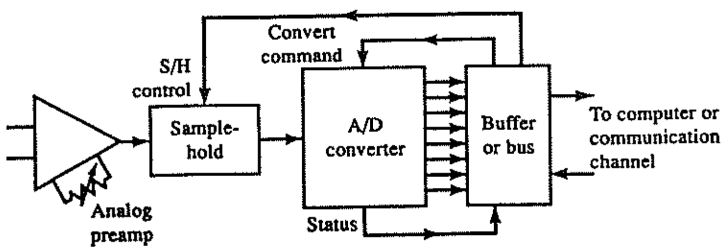
Practically: filter must be simple to implement in analog hardware



* in practice we usually oversample signal so that the "sloppy" transition band doesn't hurt us.

Analog-to-Digital Conversion

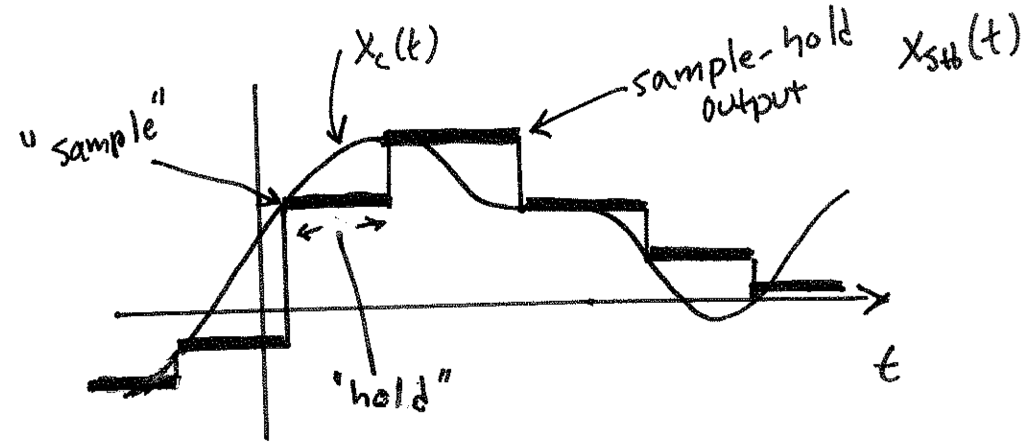




- non-ideal impulse sampling (sample-hold)

- quantization of real-valued current or voltage into a finite set of values or levels

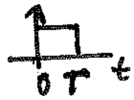
Sample-and-hold

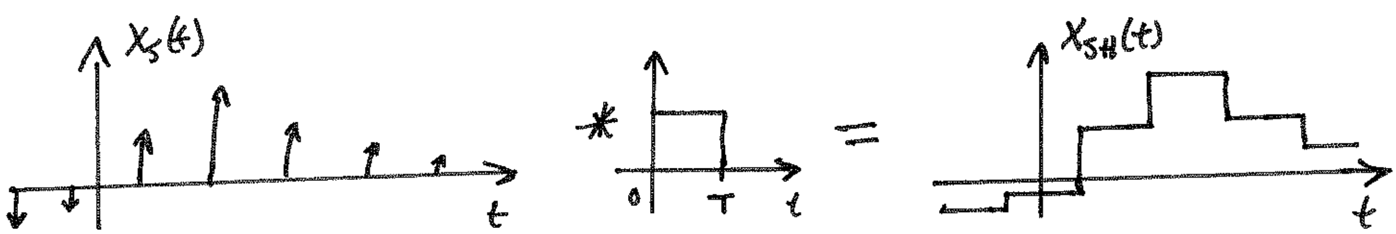


- "sample" $x_c(t)$ once every T seconds
- "hold" voltage/current at sampled-value for T seconds

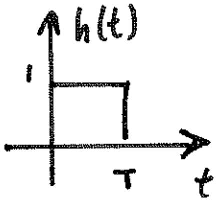
  This is necessary so that the A/D converter circuitry has time to make the conversion

Frequency Domain Interpretation:

Note sample-hold signal is effectively the ideal impulse sampled signal $x_s(t)$ convolved with  — a short pulse of width T



. . . in frequency



$$h(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{o.w.} \end{cases}$$

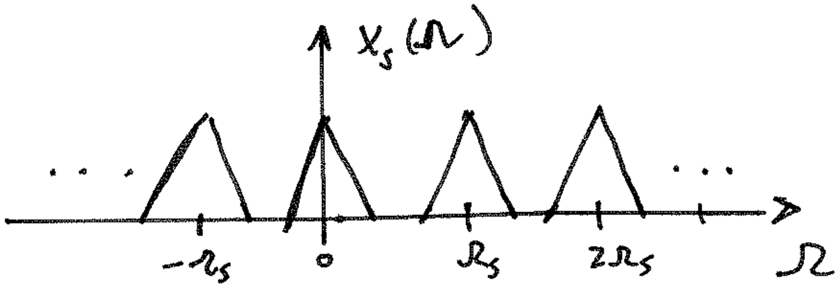
CTFT $\longleftrightarrow$

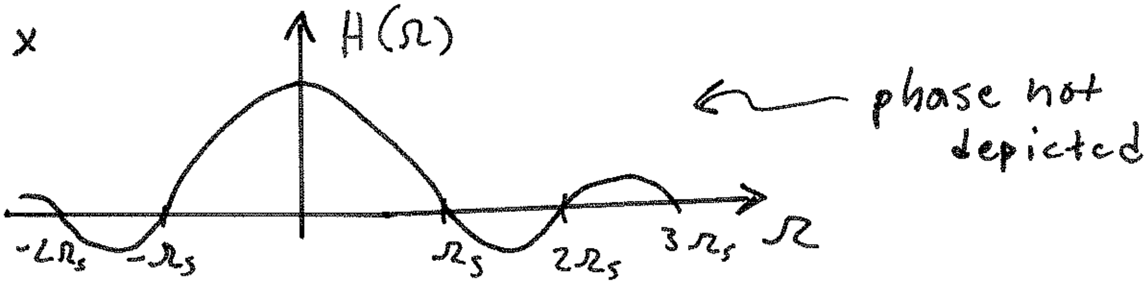
$$H(\Omega) = \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} dt$$

$$= \int_0^T e^{-j\Omega t} dt$$

$$= e^{-j\frac{\Omega}{2}t} \underbrace{\frac{\sin \frac{\Omega}{2} T}{\frac{\Omega}{2}}}_{\text{sinc}}$$

So,





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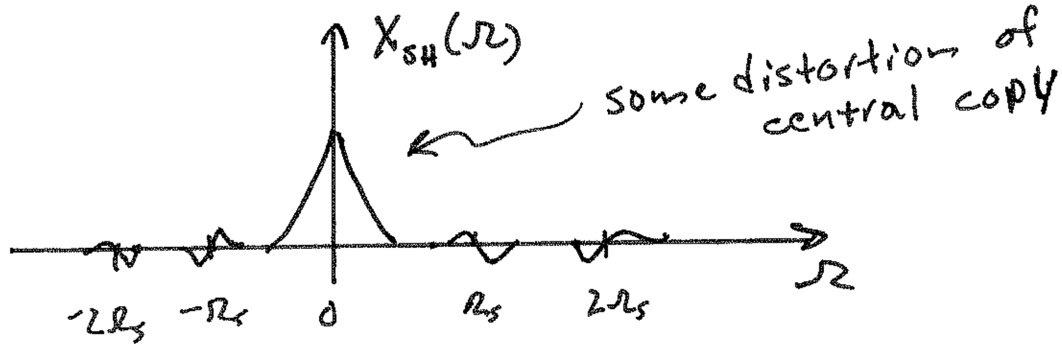
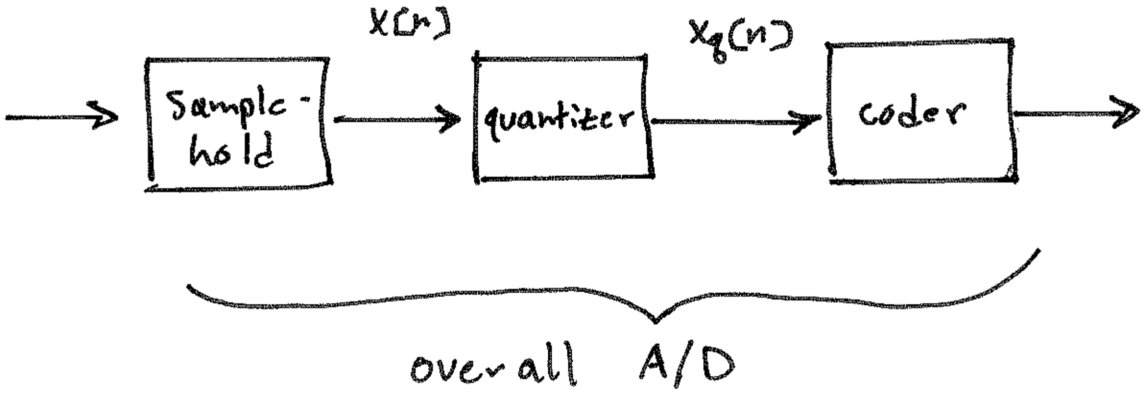


How can this distortion be minimized?

Quantization & Coding



Let $x[n] \equiv x(nT)$ as before.

Suppose that $x[n]$'s amplitude range is known (and limited).

Divide signal's amplitude range into L intervals:

$$I_k = \{ a_k < x[n] < a_{k+1} \}$$

$$k = 1, 2, \ldots, L$$

where $a_1, \ldots, a_{L+1}$ are the <u>decision levels</u>.

The possible outputs of the quantizer (<u>quantization levels</u>) are denoted

$\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_L$.

The operation of the quantizer is defined as

$$x_q[n] = Q(x[n]) = \hat{x}_k$$

if $x[n] \in I_k$

(i.e., $x_k < x[n] \le x_{k+1}$)

In most DSP systems

1. <u>Q is memoryless</u>
   - quantization of $x(n)$ depends only on that sample, no others
   - quantization operation is independent of n (time-invariant)

2. <u>Q is uniform</u>

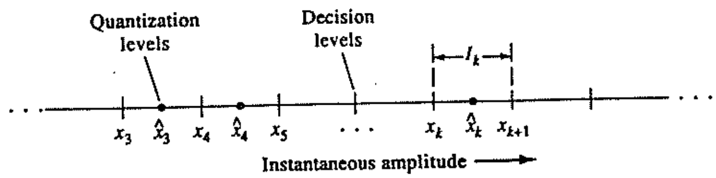
$$\hat{x}_{k+1} - \hat{x}_k = \Delta$$

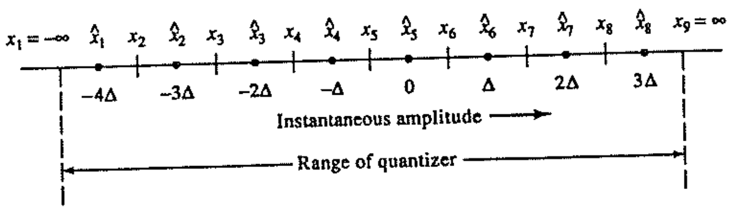
$$x_{k+1} - x_k = \Delta$$

"uniform quantizer"

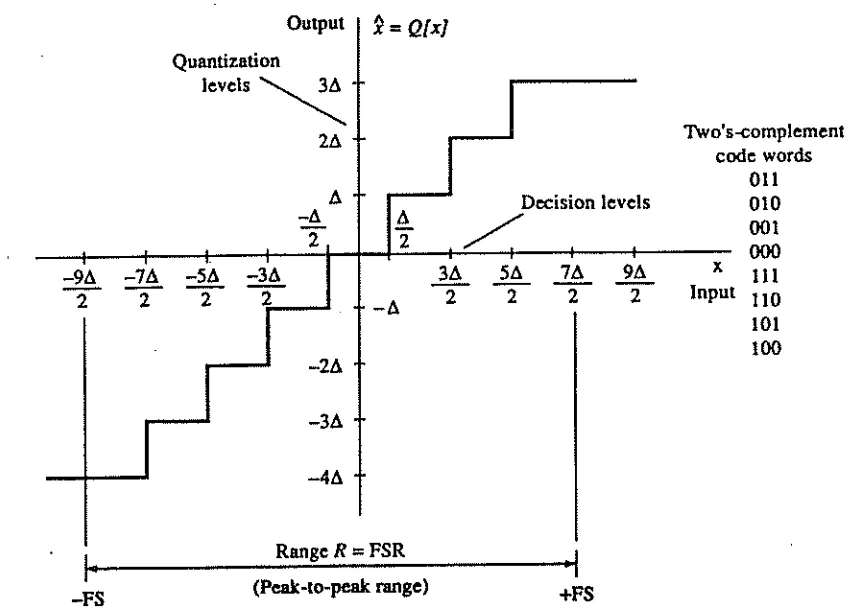
$\boxed{\Delta = \text{quantization step size}}$

Quantization process:





I/O Relationship:



Quantization Error:

$$e_q[n] = x[n] - x_q[n]$$

Note

$$-\frac{\Delta}{2} < e_q[n] \leq \frac{\Delta}{2}$$

Coding:

Assigns a unique binary number to each quantization level.

If we have L levels, then we need at least L different binary numbers.

Using $b+1$ bit word-length provides us with 2^{b+1} binary numbers or codes.

Hence, we require

$$2^{b+1} \geq L \qquad \text{or} \qquad b+1 \geq \log_2 L$$

With a (b+1) length coding scheme, the step size or resolution of the A/D converter is

$$\Delta = \frac{R}{2^{b+1}}$$

where R is the range of the quantizer.

Commonly Used Bipolar Codes

	Decimal Fraction					
Number	**Positive Reference**	**Negative Reference**	**Sign + Magnitude**	**Two's Complement**	**Offset Binary**	**One's Complement**
+7	$+\frac{7}{8}$	$-\frac{7}{8}$	0 1 1 1	0 1 1 1	1 1 1 1	0 1 1 1
+6	$+\frac{6}{8}$	$-\frac{6}{8}$	0 1 1 0	0 1 1 0	1 1 1 0	0 1 1 0
+5	$+\frac{5}{8}$	$-\frac{5}{8}$	0 1 0 1	0 1 0 1	1 1 0 1	0 1 0 1
+4	$+\frac{4}{8}$	$-\frac{4}{8}$	0 1 0 0	0 1 0 0	1 1 0 0	0 1 0 0
+3	$+\frac{3}{8}$	$-\frac{3}{8}$	0 0 1 1	0 0 1 1	1 0 1 1	0 0 1 1
+2	$+\frac{2}{8}$	$-\frac{2}{8}$	0 0 1 0	0 0 1 0	1 0 1 0	0 0 1 0
+1	$+\frac{1}{8}$	$-\frac{1}{8}$	0 0 0 1	0 0 0 1	1 0 0 1	0 0 0 1
0	0+	0−	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0
0	0−	0+	1 0 0 0	(0 0 0 0)	(1 0 0 0)	1 1 1 1
−1	$-\frac{1}{8}$	$+\frac{1}{8}$	1 0 0 1	1 1 1 1	0 1 1 1	1 1 1 0
−2	$-\frac{2}{8}$	$+\frac{2}{8}$	1 0 1 0	1 1 1 0	0 1 1 0	1 1 0 1
−3	$-\frac{3}{8}$	$+\frac{3}{8}$	1 0 1 1	1 1 0 1	0 1 0 1	1 1 0 0
−4	$-\frac{4}{8}$	$+\frac{4}{8}$	1 1 0 0	1 1 0 0	0 1 0 0	1 0 1 1
−5	$-\frac{5}{8}$	$+\frac{5}{8}$	1 1 0 1	1 0 1 1	0 0 1 1	1 0 1 0
−6	$-\frac{6}{8}$	$+\frac{6}{8}$	1 1 1 0	1 0 1 0	0 0 1 0	1 0 0 1
−7	$-\frac{7}{8}$	$+\frac{7}{8}$	1 1 1 1	1 0 0 1	0 0 0 1	1 0 0 0
−8	$-\frac{8}{8}$	$+\frac{8}{8}$		(1 0 0 0)	(0 0 0 0)	

Two's complement is the most common representation in most processors, so it's convenient to adopt this coding scheme in A/D conversion.

Two's Complement Representation

$(b+1)$-bit code : $\beta_0 \beta_1 \cdots \beta$ ← LSB = least significant bit

Value: $-\beta_0 2^0 + \beta_1 2^{-1} + \beta_2 2^{-2} + \cdots + \beta_b 2^{-b}$

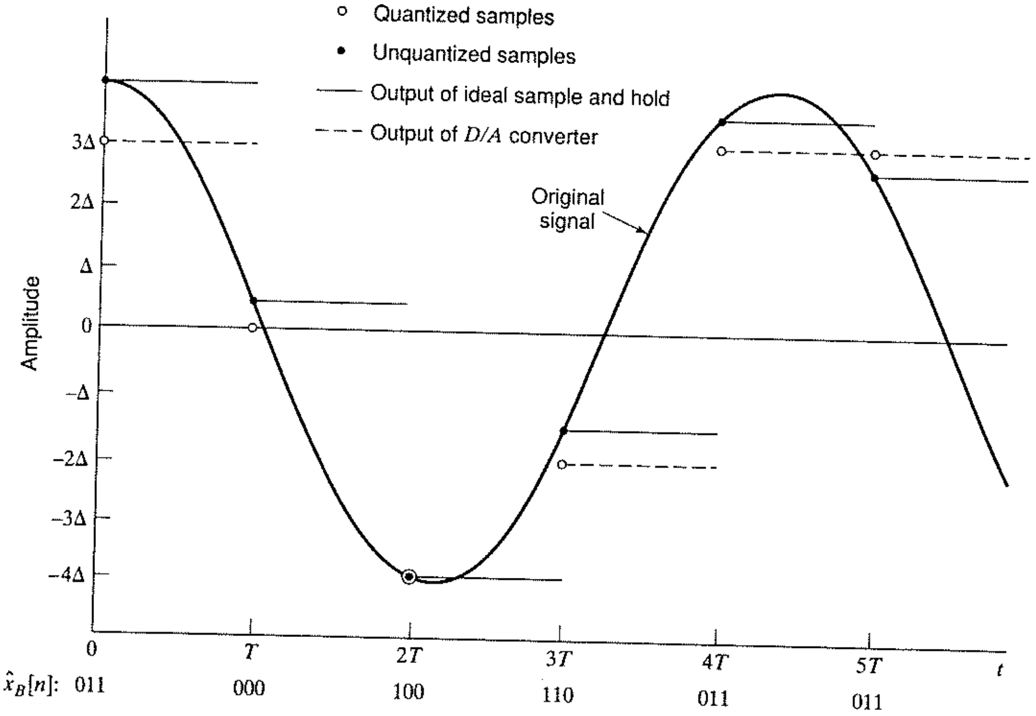


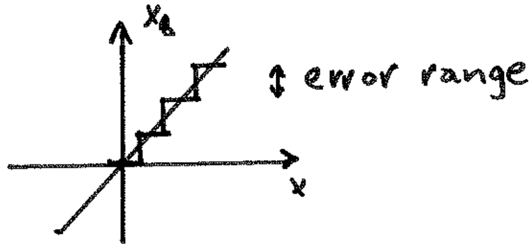
Figure 4.49 Sampling, quantization, coding, and D/A conversion with a 3-bit quantizer.

Analysis of Quantization Errors

Worst case: $\max |\underbrace{x[n] - x_q[n]}_{e_q[n]}| = \frac{\Delta}{2}$

Most errors fall somewhere in between

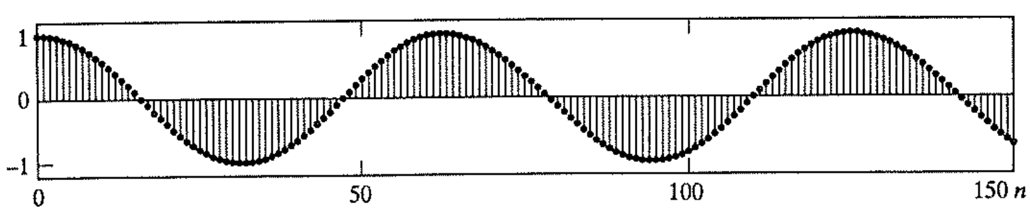
$$0 \le |e_q[n]| \le \frac{\Delta}{2}$$



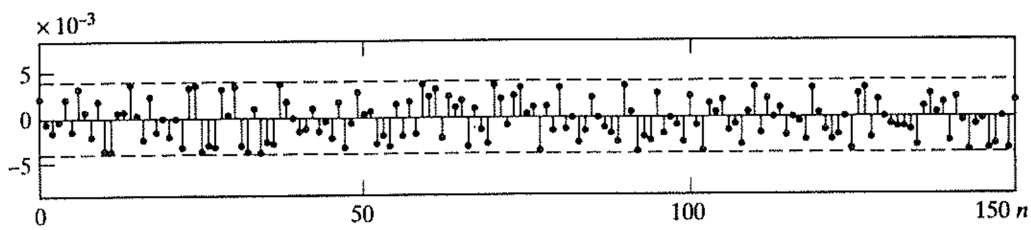
The error between $x[n]$ and $x_q[n]$ is referred to as <u>granular noise</u>.

Ex.

Sinusoidal input $x[n]$



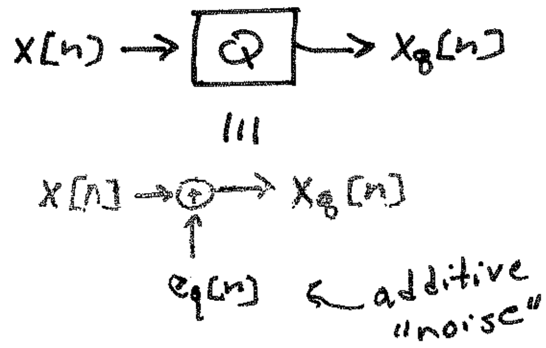
$e_q[n] = x[n] - x_q[n]$



Statistical Analysis of Granular Noise

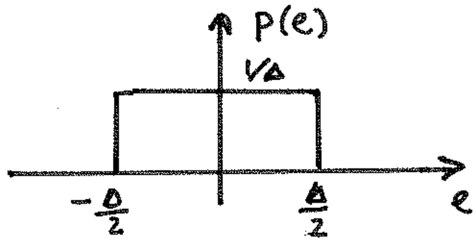
Because, of the complicated nature of quantization errors, it can be useful to regard the granular noise as a random process (instead of a deterministic one).

Is this reasonable?



Assumptions:

1. $e_q[n]$ is a realization of a stationary (wide-sense) random process
2. $e_q[n]$ is uniformly distributed over $(-\frac{\Delta}{2}, \frac{\Delta}{2}]$



and zero-mean

3. $e_q[n]$ is an uncorrelated process (white noise)
4. $e_q[n]$ is independent of $x[n]$, also a WSS process.

★ These assumptions do not hold, in general. However, if the quantization step size Δ is small enough, then the signal sequence $x(n)$ traverses several quantization levels between successive samples and 1-4 are reasonable.

Signal-to-Quantization Noise Ratio (SQNR)

$$SQNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2} \qquad (\text{dB})$$

$$\sigma_x^2 = E\left[x^2[n] \right]$$

$$\sigma_e^2 = E\left[e_Q^2[n] \right]$$

$$= \int_{-\frac{\Delta}{2}}^{\Delta/2} e^2 \, p(e) \, de$$

$$SQNR = 10 \log_{10} \left(12 \frac{\sigma_x^2}{\Delta^2} \right)$$

Now recall,

$$\Delta = \frac{R}{2^{b+1}}$$

$$SQNR = 10 \log_{10}\left(12 \frac{\sigma_x^2 (2^{b+1})^2}{R^2}\right)$$

$$= 20 \log_{10}(2^{b+1}) + 10 \log_{10}(12) - 20 \log_{10}\left(\frac{R}{\sigma_x}\right)$$

$$= 6.02\, b + 16.81 - 20 \log_{10}\left(\frac{R}{\sigma_x}\right)$$

depends on R (range) and statistics of $x[n]$

Suppose that $x[n]$ is a stationary Gaussian process and that the quantizer range is

$$R = [-3\sigma_x, +3\sigma_x].$$

Then less than 3 out of every 1000 samples would fall out of the range on average, and

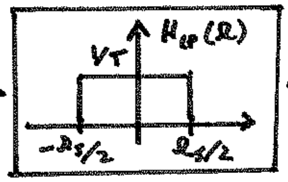
$$\boxed{SQNR \simeq 6.02b + 1.25 \text{ dB}}$$

commonly used to specify precision needed in A/D converters

Digital-to-Analog Conversion

Ideal:

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

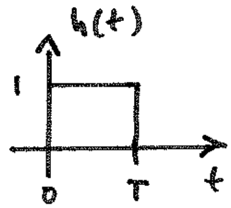
$$y[n] \longrightarrow \sum_{n=-\infty}^{\infty} y[n]\,\delta(t-nT) \longrightarrow$$  $$\longrightarrow y_r(t)$$

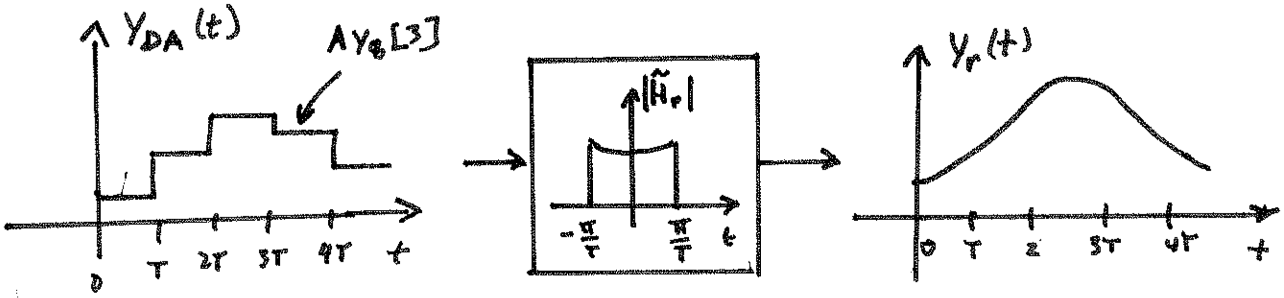
Real-World Reconstruction Using Sample-Hold:

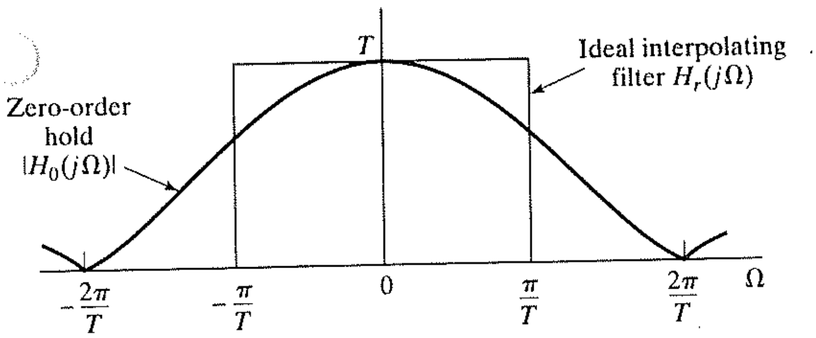
$y_Q[n] \longrightarrow$ [Scale by A] $\longrightarrow$ [sample and hold] (H) $\xrightarrow{y_{DA}(t)}$ [lowpass smoothing filter] ($\tilde{H}_r$) $\longrightarrow y_r(t)$

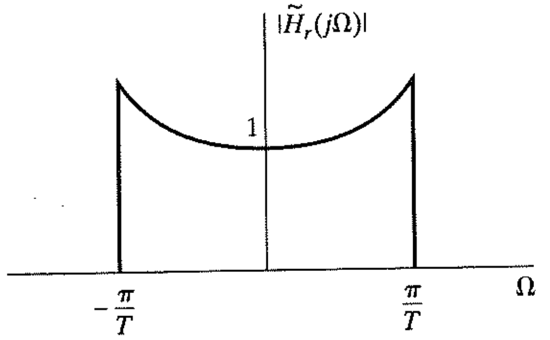
$\underbrace{\qquad\qquad\qquad\qquad}_{\text{real-world D/A}}$

$$y_{DA}(t) = \sum_{n=-\infty}^{\infty} A\, y_Q[n]\, h(t-nT) \quad ,$$

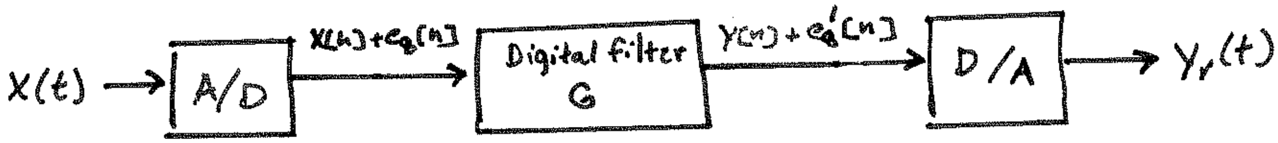








DSP System Analysis



$$e_q'[n] = \sum_{k=-\infty}^{\infty} g[k]\, e_q[n-k]$$

$$S_{ee}(\omega) =$$

$$\Rightarrow \quad S_{e'e'}(\omega) =$$

SET F: The z-transform

OSB Chapter 3

Outline

I. z-transform

II. Properties and ROC

III. Inverse z-Transform

IV. z-Transform Analysis of LTI Systems and Filters

A Brief intro to filter design

An FIR (finite impulse response) filter takes the form

$$x[n] \longrightarrow \boxed{\overset{H}{\text{FIR}}} \longrightarrow y[n]$$

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k]$$

The frequency response of an FIR filter is given by its DTFT

$$H(\omega) = \sum_{k=0}^{M-1} h[k]\, e^{-j\omega k}$$

FIR filters are just one type of a more general class of realizable DSP filters known as linear, constant-coefficient difference equations (LCCDEs).

The general form of an LCCDE is

$$y[n] = \sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

Note that this type of filter is IIR (infinite impulse response) in general, but also includes FIR filters ($a_k = 0$, $k = 1, \ldots, N$).

What is the frequency response of an LCCDE?

$$y[n] = \sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

$$Y(\omega) =$$

I. The z-Transform

<u>What is it?</u>

1. Counterpart of the LAPLACE TRANSFORM for DT signals and systems

2. A Generalization of the DTFT

<u>Why generalize the DTFT?</u>

- Because the DTFT doesn't exist (i.e., converge) for many important signals.

- Notationally cleaner than DTFT (easy to work with).

- Brings the power of complex variable theory to bear on DSP.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \quad z \in \mathbb{C}}$$

set of complex numbers

The Z-Transform

A generalization of the DTFT:

DTFT:
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Complex number
$|e^{-j\omega n}| = 1$

Z-transform:
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

general complex number
$|z| \in \mathbb{R}$

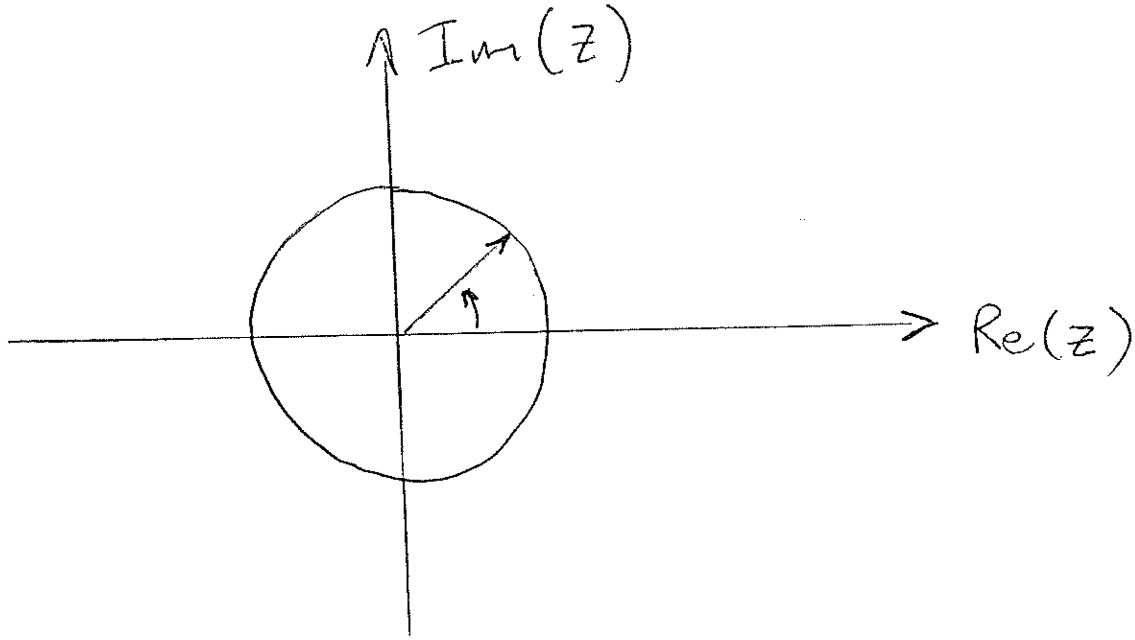
Note:

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

☆ DTFT is a complex-valued function of a real-valued variable

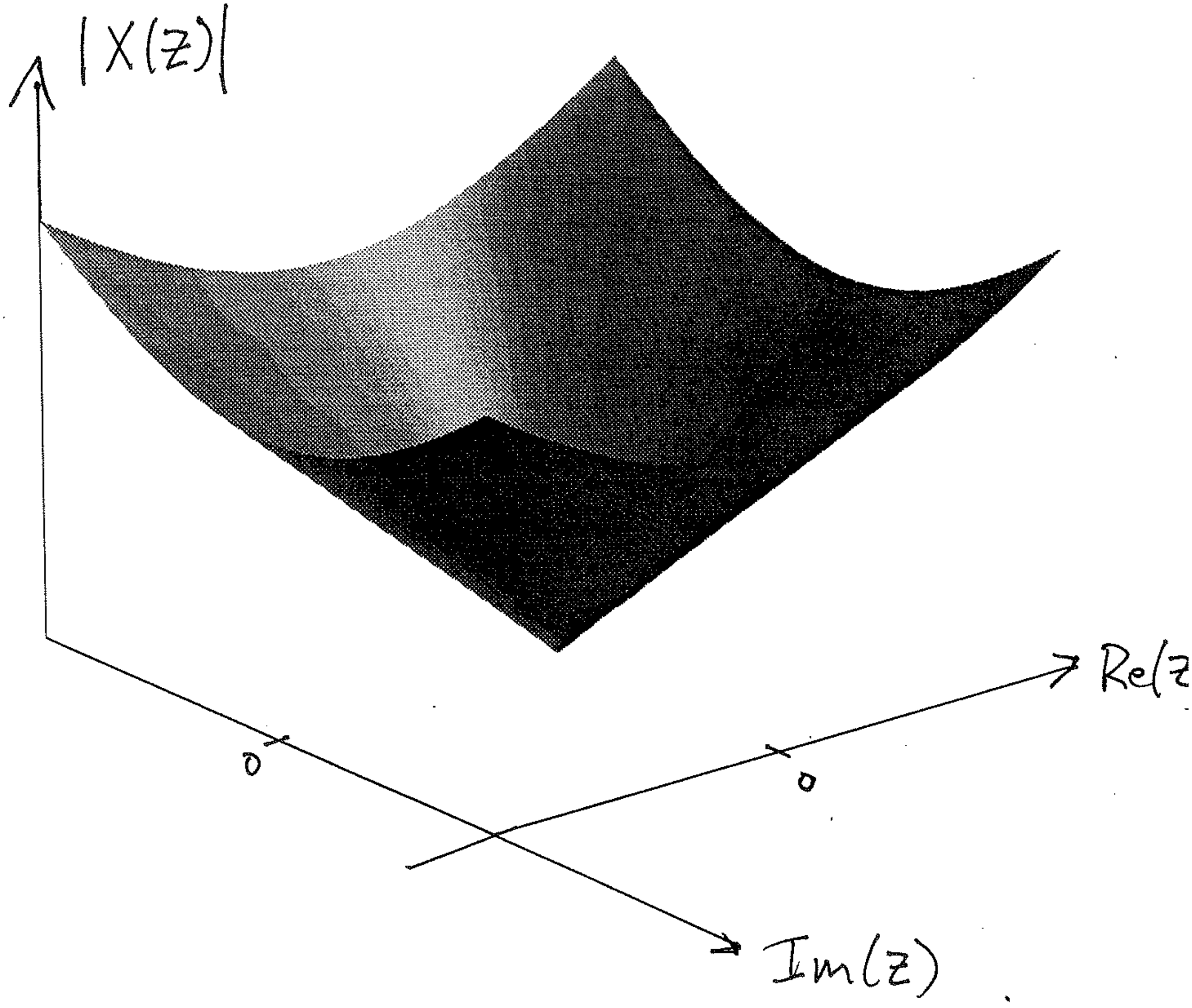
☆ Z-Transform is a complex-valued function of a complex-valued variable z.

The Complex Plane (The z-plane)



- $X(z)$ is defined everywhere on this plane
- $X(\omega)$ is defined only on the <u>unit circle</u>

ex. $X(z) = z$



It's a cone!

What is the DTFT in this case?

Ex. Sketch $X(z) = \dfrac{z}{z-a}$

Note: The DTFT corresponds to the evaluation of the z-Transform on the unit circle

$\Rightarrow$ periodicity of $X(\omega)$ becomes obvious

We write:

$$\mathcal{Z}\{x[n]\} = X(z)$$ "z-Transform operator"

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

Ex. $\boxed{x[n] = \delta[n-n_0]}$ delay

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

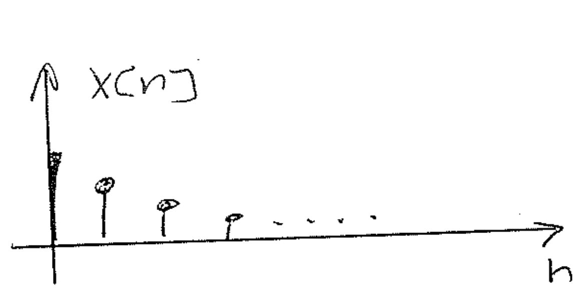
$$= \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$$

$$\boxed{X(z) = z^{-n_0}}$$

$n_0 = 1 \longrightarrow X(z) = z^{-1}$ (unit delay in z-Transfo Domain

Example : Let's look at

$$x[n] = \alpha^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} \qquad \left(\text{if } |\alpha z^{-1}| < 1 \right)$$

$$\boxed{X(z) = \frac{z}{z-\alpha}}$$

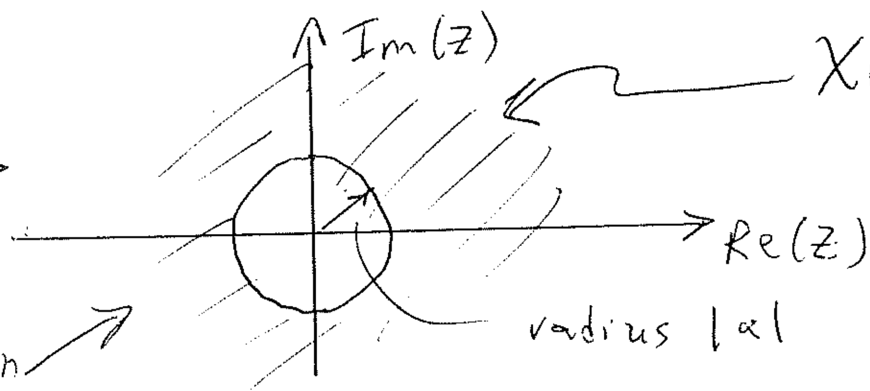
What if $|\alpha z^{-1}| \geq 1$?

Then $\sum_{n=0}^{\infty} (\alpha z^{-1})^n$ does not converge!

Therefore we must specify the set of z's for which the z-transform exists.

$$|\alpha z^{-1}| < 1 \longrightarrow |z| > |\alpha|$$

Note : $|\alpha|$ can be bigger than 1.

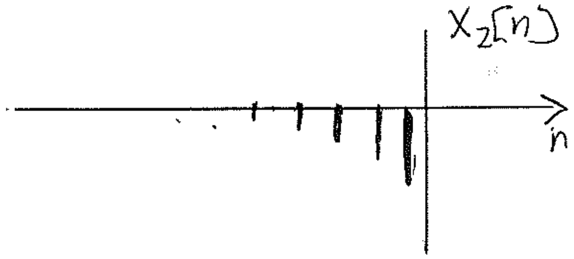


$X(z)$ exists outside the circle $|z|=|\alpha|$

Shaded region is "Region of Convergence" R.O.C.

Ex. "The evil twin"

$$x_2[n] = -\alpha^n u[-1-n] \qquad \left(\begin{array}{l} -1-n \geq 0 \\ \rightarrow n \leq -1 \end{array}\right)$$



$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -(\alpha^{-1} z)^{-n}$$

$$= -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n$$

$$|\alpha^{-1} z| < 1 \longrightarrow X_2(z) = -\sum_{n=0}^{\infty} (\alpha^{-1} z)^n + 1$$

$$= \frac{1}{-1 + \alpha^{-1} z} + 1$$

$$= \frac{\alpha}{z - \alpha} + 1$$

Non-uniqueness of z-transforms!

$$= \frac{\alpha + z - \alpha}{z - \alpha} = \frac{z}{z - \alpha}$$

Region of Convergence

Clearly The region where $X(z)$ exists is pretty important.

Defn: For a given signal $x[n]$, the range of z for which the z-transform converges is called the ROC.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ is a power series that converges when $x[n] z^{-n}$ is abs. summ.

ie,

$$\sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty$$

e.g. ROCs of $\mathcal{Z}\{\alpha^n u[n]\}$ and $\mathcal{Z}\{-\alpha^n u[-1-n]\}$ are drastically different.

Poles & Zeros

Among the most useful & important z-transforms are

Rational Functions $\quad X(z) = \dfrac{P(z)}{Q(z)}$

with $P(z)$ and $Q(z)$ polynomials in z

Zeros: Values of z for which $X(z) = 0$
$(P(z) = 0)$

Poles: Values of z for which $X(z) = \infty$
$(Q(z) = 0)$

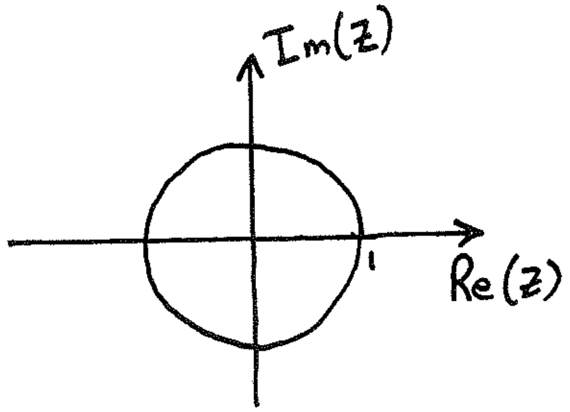
- For finite values of z, poles are the roots of $Q(z)$
- In addition, poles may occur at $z = 0$ or $z = \infty$

We denote poles in a z-plane plot by "x"
We denote zeros by "o"

* Clearly the ROC cannot contain any poles. Why?

<u>Ex.</u> $x_1[n] = \alpha^n u[n] \overset{Z}{\longleftrightarrow} X_1(z) = \dfrac{z}{z-\alpha}$

$|z| > |\alpha|, \quad |\alpha| < 1$

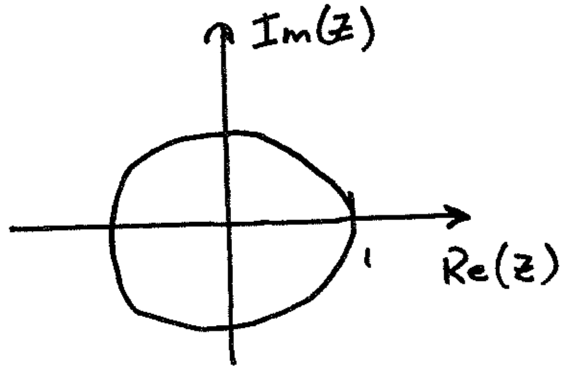


zero:

pole:

$x_2[n] = -\alpha^n u[-1-n] \overset{Z}{\longleftrightarrow} X_2(z) = \dfrac{z}{z-\alpha}$

$|z| < |\alpha| < 1$



zero:

pole:

Note: Both $X_1(z)$ & $X_2(z)$ are <u>rational</u>

Note: the z-transform is <u>linear</u>:

$y[n] = x_1[n] + x_2[n]$

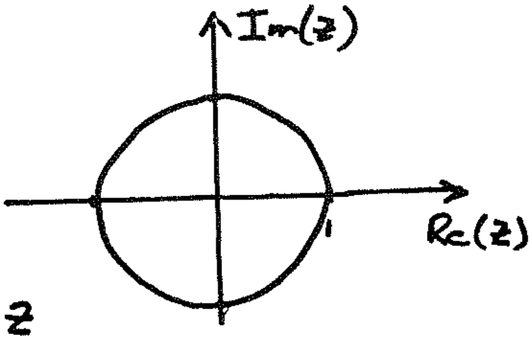
$Y(z) = \sum_n y[n] z^{-n} =$

$=$

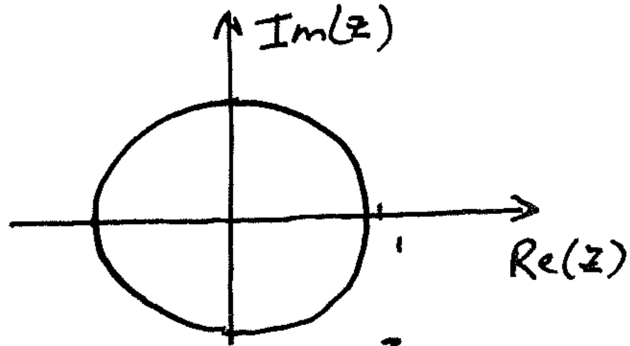
Ex. $x_3[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\;Z\;} \frac{z}{z-\frac{1}{2}}$ ROC: ______________

$\left(-\frac{1}{3}\right)^n u[n] \xleftrightarrow{\;Z\;} \frac{z}{z+\frac{1}{3}}$ ROC: ______________



$\frac{z}{z-\frac{1}{2}}$



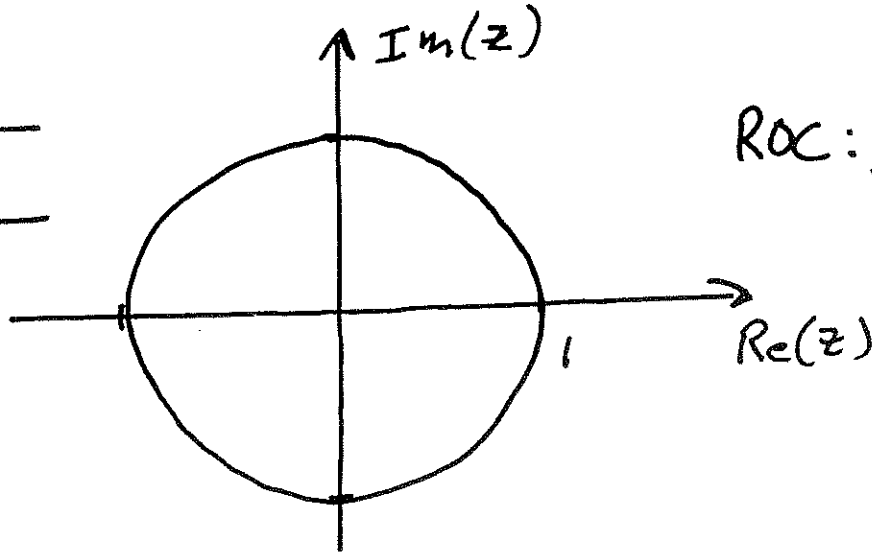
$\frac{z}{z+\frac{1}{3}}$

Linearity $\Rightarrow$ $X_3(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{z\left(2z-\frac{1}{6}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$

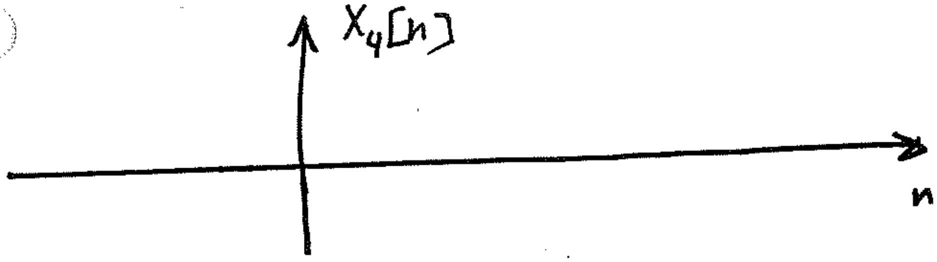
Zeros: ______________

Poles: ______________

ROC: __________

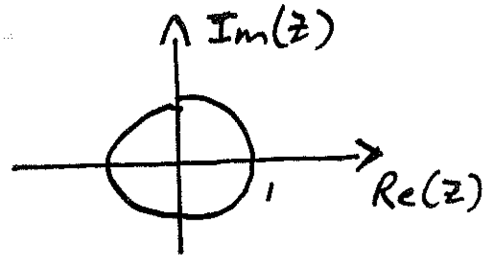


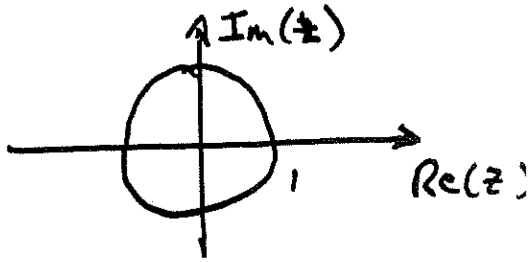
Ex. $x_4[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$



$\left(-\frac{1}{3}\right)^n u[n] \overset{Z}{\longleftrightarrow}$

$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow}$

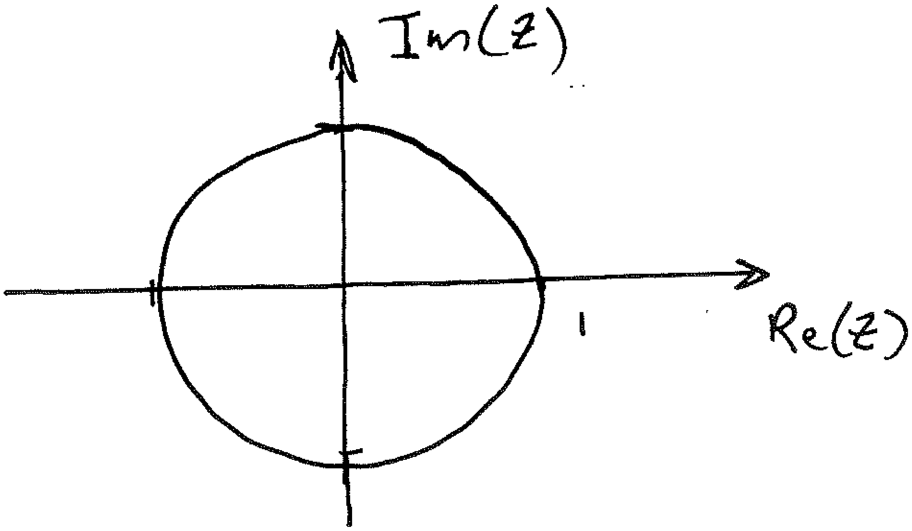




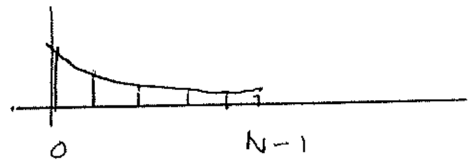
So, by linearity

$X_4(z) =$

ROC: ______________



Ex. $x[n] = \begin{cases} \alpha^n, & 0 \le n \le N-1 \\ 0, & \text{o.w.} \end{cases}$



$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n} = \sum_{n=0}^{N-1} (\alpha z^{-1})^n$$

$$= \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - \alpha^N}{z-\alpha}$$

ROC: $\sum_{n=0}^{N-1} |\alpha z^{-1}|^n < \infty \Longleftrightarrow |\alpha| < \infty, \ \& \ |z| \neq 0$

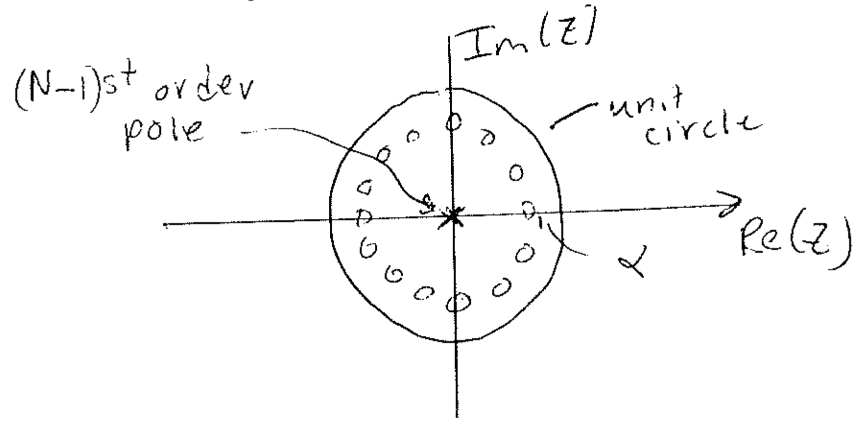
So, if $|\alpha| < \infty$, ROC is the entire z-plane except origin.

Zeros: $z^N = \alpha^N \longrightarrow z_k = \alpha e^{j\frac{2\pi}{N}k}$

$k = 0, 1, \ldots, N-1$

"N^{th} root of unity"

poles: $z = 0$ (N-1 of them) + $\cancel{z=\alpha}$ cancelled by a zero at α



$\boxed{\text{Inverse z-transform}}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

<u>Question:</u> Given $X(z)$, how can we find $x[n]$?

Inverting the z-transform is different from inverting, say, the DTFT.

We'll look at four methods:

$\left.\begin{array}{l}\text{a. Inspection} \\ \text{b. Partial Fraction Expansion}\end{array}\right\}$ "tables"

c. Power series Expansion

d. Contour Integration

(a) Inspection

Basically, become familiar with the z-transform pair tables and "reverse engineer"

ex. $X(z) = \frac{z}{z-a}$, $|z| > |a|$

"looks mighty familiar"

Looking in tables gives $x[n] = a^n u[n]$.

(b) Partial Fractions Expansion

Basically this: Break a rational $X(z)$ up into elementary forms, each of which can be inverse transformed "by inspection"

Ex. $$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{8}{1 - z^{-1}} - \frac{9}{1 - \frac{1}{2}z^{-1}} + 2$$

$|z| > 1$

easy to invert!

PF expansion

☆ PF expansion always works for rational functions

☆☆ Review / Relearn how to do PF

Partial Fractions

Given $$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$ rational in z^{-1}

Note: We can express this as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$\{c_k\}$ - nonzero zeros of $X(z)$

$\{d_k\}$ - nonzero poles of $X(z)$

We would like to express $X(z)$ as a sum in terms of the form

$$\frac{A_k}{1 - d_k z^{-1}}$$ ← can be inverted "by inspection"

STEP I: If $M \geq N$, divide numerator by denominator to get

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \frac{\sum_{k=0}^{N-1} b'_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

eg. $$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$ $M = N = 2$

$\Rightarrow$ Have to divide

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{-z^{-2} + 3z^{-1} - 2} \\ & \qquad 5z^{-1} - 1 \end{array}$$

long division

$$\Longrightarrow X(z) = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

STEP 2: Factor denominator

ex. $$X(z) = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

Factor: $(1 - \frac{1}{2}z^{-1})(1 - z^{-1})$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

STEP 3: Break $X(z)$ into sum of factors of the form

$$\frac{A_k}{1 - d_k z^{-1}}, \quad \sum_{m=1}^{s} \frac{C_m}{(1 - d_k z^{-1})^m}$$

1st order (non-repeated)

s-th order pole (repeated s times)

* re-learn how to deal with this *

ex. $$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$= 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$$\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{1-z^{-1}}$$

$$= \frac{A_1(1-z^{-1}) + A_2(1-\frac{1}{2}z^{-1})}{(1-\frac{z^{-1}}{2})(1-z^{-1})}$$

$$\Longrightarrow \quad -1+5z^{-1} = A_1(1-z^{-1}) + A_2(1-\tfrac{1}{2}z^{-1})$$

<u>For A_1:</u> $z^{-1} = 2 \Longrightarrow A_1(-1) = 9$

$$\boxed{A_1 = -9}$$

<u>For A_2:</u> $z^{-1} = 1 \Longrightarrow A_2(1-\frac{1}{2}) = -1+5$

$$\boxed{A_2 = 8}$$

$$\boxed{X(z) = 2 + \frac{-9}{(1-\frac{1}{2}z^{-1})} + \frac{8}{1-z^{-1}}}$$

<u>STEP 4:</u> Invert, by inspection, each term (use tables)

$$\begin{cases} \dfrac{1}{1-az^{-1}}, \quad |z|>a, \quad \overset{Z}{\longleftrightarrow} \quad a^n u[n] \\[2ex] \dfrac{1}{1-az^{-1}}, \quad |z|<a, \quad \overset{Z}{\longleftrightarrow} \quad -a^n u[-1-n] \end{cases}$$

(<u>Note:</u> $\frac{1}{1-az^{-1}} = \frac{z}{z-a}$)

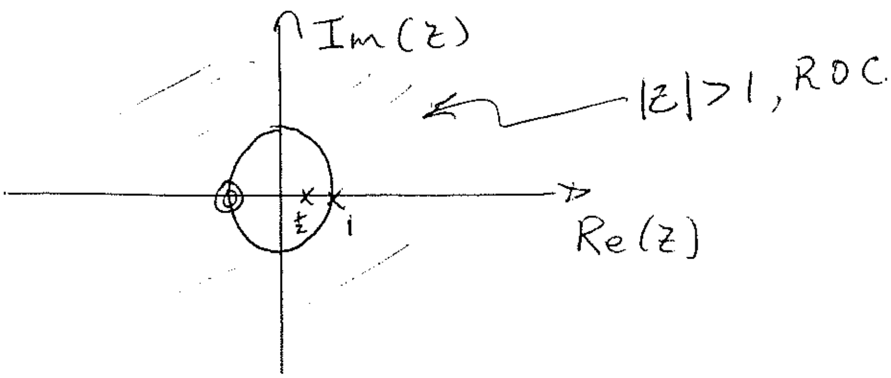
ex. $X(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$, $|z| > 1$

$$= 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

$$\updownarrow z \qquad \updownarrow z \qquad \updownarrow z$$

$$\boxed{x[n] = 2\delta[n] - 8\left(\tfrac{1}{2}\right)^n u[n] + 8u[n]}$$

Suppose $|z| > 1$ was not specified. What other ROC's are possible?



Other possibilities:

(1) $|z| < \frac{1}{2}$

(2) $\frac{1}{2} < |z| < 1$

What are the time-domain signals for (1) and (2)?

(c) Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots x[-2] z^{2} + x[-1] z + x[0] + x[1] z^{-1} + \cdots$$

inverse transform comes up as <u>coefficients</u> of Laurent series expansion of $X(z)$

☆ <u>Great</u> for finite-length signals ☆

<u>ex.</u> $X(z) = z^{2}\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$

$$= z^{2} - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

☆ <u>Useful</u> when $X(z)$ is not <u>rational</u> ☆

ex $X(z) = \log(1 + a z^{-1}) \qquad |z| > |a|$

complex logarithm

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

$$X(z) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n} a^n}_{x[n]} z^{-n}$$

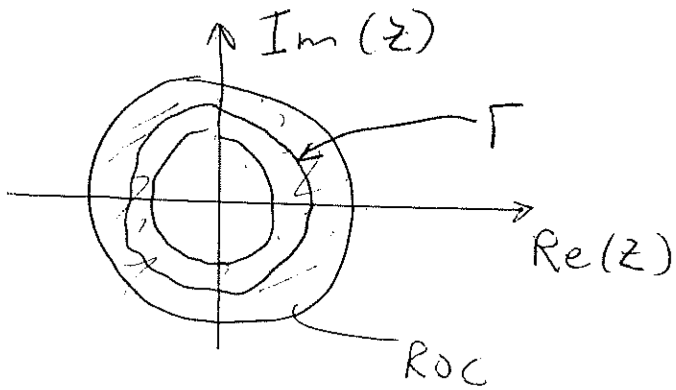
$$\Rightarrow \quad x[n] = \begin{cases} \frac{(-1)^{n+1}}{n} a^n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

(d) Inverse z-transform by Contour Integration

Without going into any details, we have

$$x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z)\, z^{n-1}\, dz$$

Where Γ is a counter-clockwise contour in the ROC of $X(z)$ encircling the origin of the z-plane



Requires knowledge of complex variable theory (Cauchy Residue Theorem)

For more details see

Digital Signal Processing
by Oppenheim and Schafer (1st edition)
p. 181-182
(if you're interested)

Z-transform Analysis of LTI Systems

Analysis of DT Systems and Filters Using the Z-Transform

Remember <u>Difference Equations</u>?

Many LTI DT systems have input-output relationships that satisfy a linear constant-coefficient DE of the form:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (1)$$

with zero initial conditions.

Take (1) to the z-domain by Z-transform of both sides

$$\mathcal{Z}\left\{ \sum_{k=0}^{N} a_k y[n-k] \right\} = \mathcal{Z}\left\{ \sum_{k=0}^{M} b_k x[n-k] \right\} \qquad (2)$$

Recall $\quad x[n-k] \xleftrightarrow{\mathcal{Z}} z^{-k} X(z)$

$\Longrightarrow$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k} \qquad (3)$$

Questions:

1. When is a difference equation **stable**?

2. When is the corresponding impulse response / system **causal**?

3. Suppose the difference equation is a model for some system that we want to "undo". Does an "**inverse system**" exist.

ex. comm signal $\to \underbrace{\boxed{H}}_{\text{channel}} \to$ distorted comm signal

$\downarrow$

$\boxed{H^{-1}}$

$\downarrow$

restored signal

4. What is the frequency response of a difference equation? How is it related to the z-transform?

$$(3) \Rightarrow \left(\sum_{k=0}^{N} a_k z^{-k} \right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) X(z)$$

$$\Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

<u>NOTES:</u>

1. $H(z)$ is a <u>rational function</u>!

2. Numerator coefficients come from RHS of (1).

3. Denominator coefficients come from LHS of (1).

<u>Ex.</u> $x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$

We are given an LTI system with transfer function

$$H(z) = \frac{(z+1)^2}{(z-\frac{1}{2})(z+\frac{3}{4})}$$

and wish to implement this system as an Linear constant-coeff. difference equation. What is the proper diff. eqn?

STABILITY and CAUSALITY

In going from

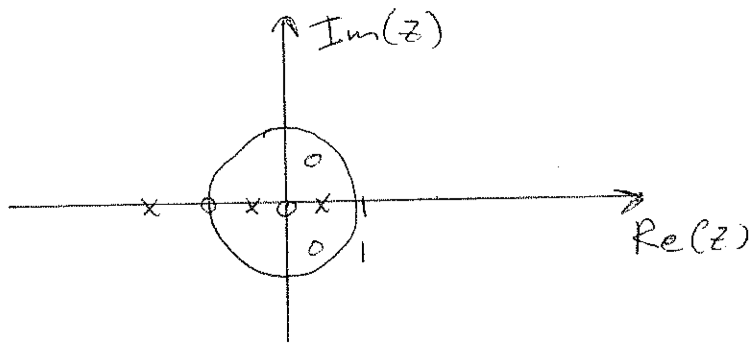
$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

to

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

we did not specify an <u>ROC</u>.

If we factor $H(z)$ into poles and zeros we can plot them in z-plane.



Several ROCs may be possible

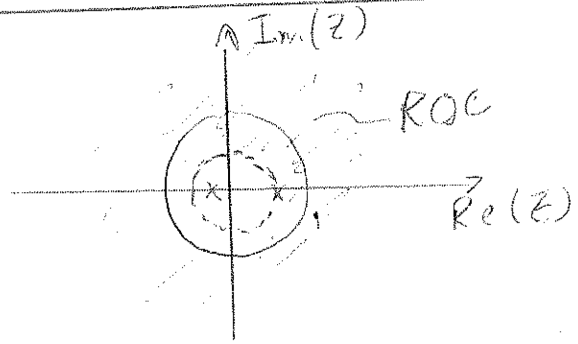
- each ROC corresponds to a different impulse response
- which one to choose?

In general, there is no "right" choice.
However, the are some choises that make sense in practice.

☆ Causality

$h[n]$ is causal if $h[n] = 0$, $n < 0$

$\Longrightarrow$ $\boxed{\text{ROC extends outward from the outermost pole.}}$



☆ Stability

A system $\rightarrow \boxed{H} \rightarrow$ is BIBO stable

Note: $$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right|$$
$$\leq \sum_{n=-\infty}^{\infty} |h[n]| \, |z^{-n}|$$

On the unit circle, i.e, $\{|z| = 1\}$

$$\rightarrow \quad |H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]| \quad , \text{for every } |z| = 1$$

Therefore,

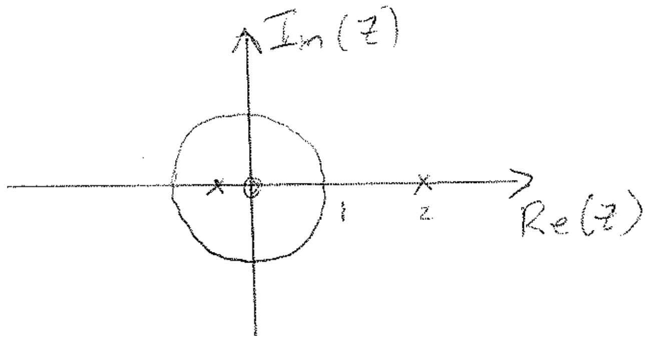
$\boxed{\text{BIBO} \Longleftrightarrow \text{ROC of } H(z) \text{ includes the unit circle}}$

Key Question:

Are stability and causality always compatible?

Ex. Consider

$$H(z) = \frac{z^2}{(z-2)(z+\frac{1}{2})}$$



and its various ROCs and corresponding inverses $h[n]$.

☆ INVERSE SYSTEMS ☆

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

After passing $x[n]$ through H to get $y[n]$, does there exist a system H_I that will give us $x[n]$ back?

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] \longrightarrow \boxed{H_I} \longrightarrow x[n]$$

Obviously we need

$$H(z)\,H_I(z) = 1$$

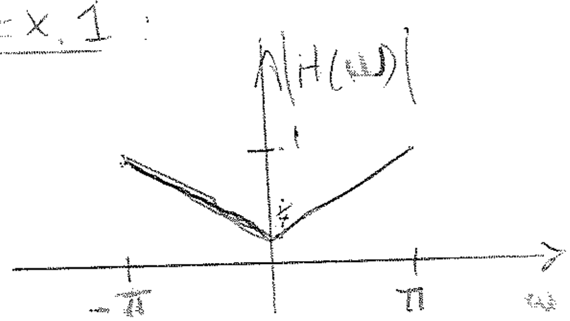
Equivalently in time domain:

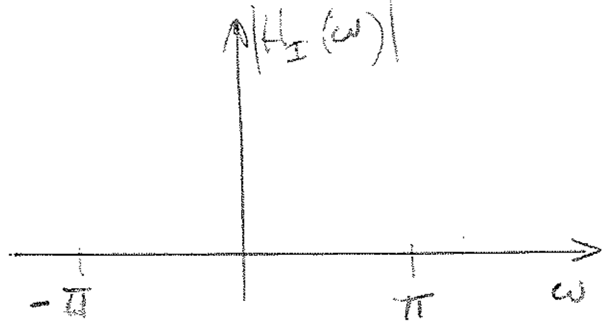
$$h[n] * h_I[n] = \underline{\delta[n]}$$

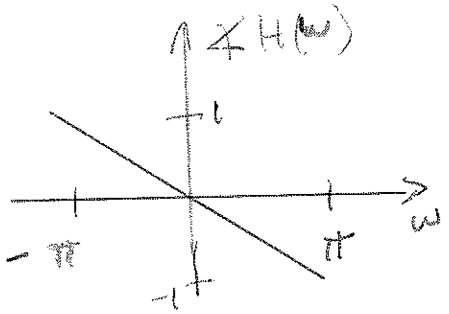
If $H_I(z)$ exists, then its frequency response is

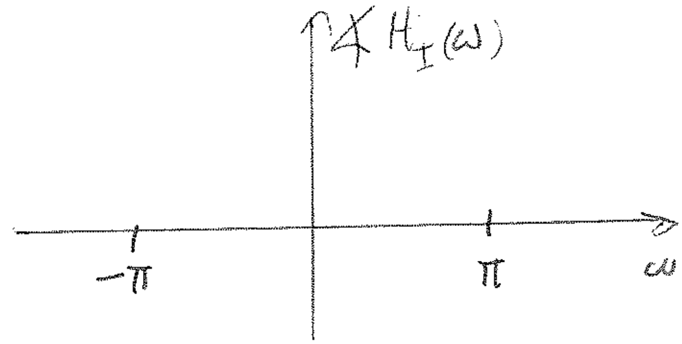
$$H_I(\omega) = \frac{1}{H(\omega)} \qquad \left(\begin{array}{c} H(z) \text{ and} \\ H_I(z) \text{ evaluated} \\ \text{on unit circle} \end{array} \right)$$

Ex. 1:

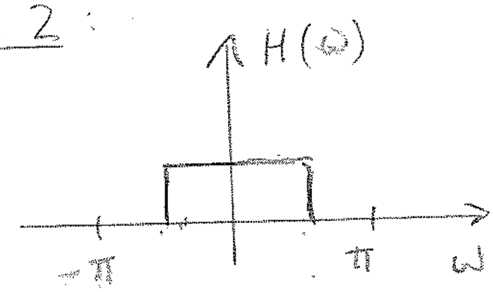


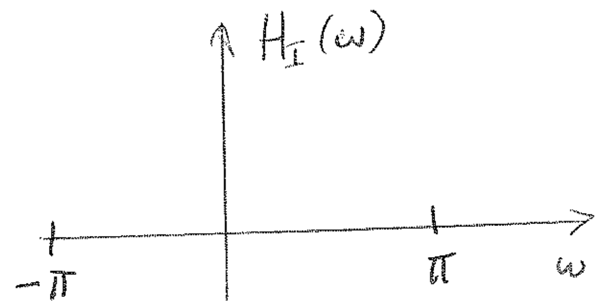






Ex. 2:





Conclusion: Inverse of BIBO system exists if and only if $|H(\omega)| > 0$, $-\pi < \omega < \pi$

Consider <u>Rational $H(z)$</u>.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

c_k's are <u>zeros</u> of $H(z)$
d_k's are <u>poles</u> of $H(z)$

$$H_I(z) = \frac{1}{H(z)} = \frac{a_0}{b_0} \frac{\prod_{k=1}^{N} (1 - d_k z^{-1})}{\prod_{k=1}^{M} (1 - c_k z^{-1})}$$

<u>poles</u>?
<u>zeros</u>?

<u>Question</u>: What is the ROC of H_I?

Recall $H_I(z) = \frac{1}{H(z)} \iff h[n] * h_I[n] = \delta[n]$

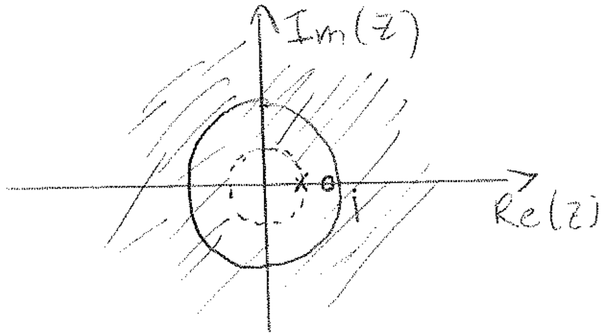
$\Downarrow$

ROC_H and ROC_{H_I} must overlap

<u>Ex.</u> $H(z) = \frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}}$, $|z| > 0.9$

Find $H_I(z)$. Is it stable? causal?
and its ROC.

$$H_I(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

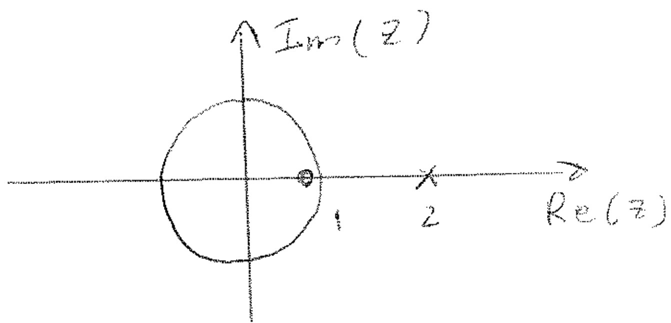


ROC . $|z| > 0.5$ <u>for causality</u>

ROC includes unit circle $\Rightarrow$ <u>stability</u>

<u>Ex.</u> $H(z) = \dfrac{z^{-1} - 0.5}{1-0.9z^{-1}}$, $|z| > 0.9$

$$H_I(z) = \frac{1-0.9z^{-1}}{z^{-1}-0.5} = -2\left(\frac{1-0.9z^{-1}}{1-2z^{-1}}\right)$$



ROC : $|z| > 2$ <u>for Causality</u>

$|z| < 2$ <u>for stability</u>

<u>Conflicting requirements</u>

Stable & Causal inverse does not exist!

(PROBLEM : H(z) has a zero <u>outside</u> the unit circle)

Notes:

(1) $H(z)$ stable, causal $\Longleftrightarrow$ poles <u>inside</u> unit circle

(2) $H(z)$ has a stable, causal inverse $H_I(z)$ $\Longleftrightarrow$ <u>zeros</u> <u>of $H(z)$ inside</u> unit circle

(1) & (2)

$H(z)$ stable & causal with stable, causal inverse if and only if all poles and zeros of $H(z)$ are <u>inside</u> the unit circle.

This type of system is called a <u>minimum phase system</u>

☆ key in important applications

Impulse Responses

$$\delta[n] \longrightarrow \boxed{H(z)} \longrightarrow \; ?$$

$$H(z) = \frac{\sum_{k=1}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}}$$

partial fraction expansion

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \underbrace{\sum_{k=1}^{N_1-1} \frac{A_k}{(1-d_k z^{-k})}}_{\text{non-repeated "1st order" poles}} + \underbrace{\sum_{k=N_1}^{N_2} \frac{C_k d_k z^{-1}}{(1-d_k z^{-1})^2}}_{\text{repeated "second-order" poles}} + \cdots \quad \ldots$$

$\updownarrow \; Z$

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N_1-1} A_k (d_k)^n u[n] + \sum_{k=N_1}^{N_2} C_k\, n (d_k)^n u[n] + \cdots$$

* Each pole contributes a term to the impulse response.

Ex.

$$H(z) = z^{-1} + \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{6}z^{-1}}{(1+\frac{1}{3}z^{-1})^2}$$

$$\Big\updownarrow$$

$h[n] =$

---

* each pole has a contribution to impulse response
* form of contribution depends on multiplicity of pole
* Behavior of impulse response contribution depends on location of pole

---

Simplest Case:

$$z^{-k} \overset{Z}{\longleftrightarrow} \delta[n-k]$$

k-th order pole @ origin

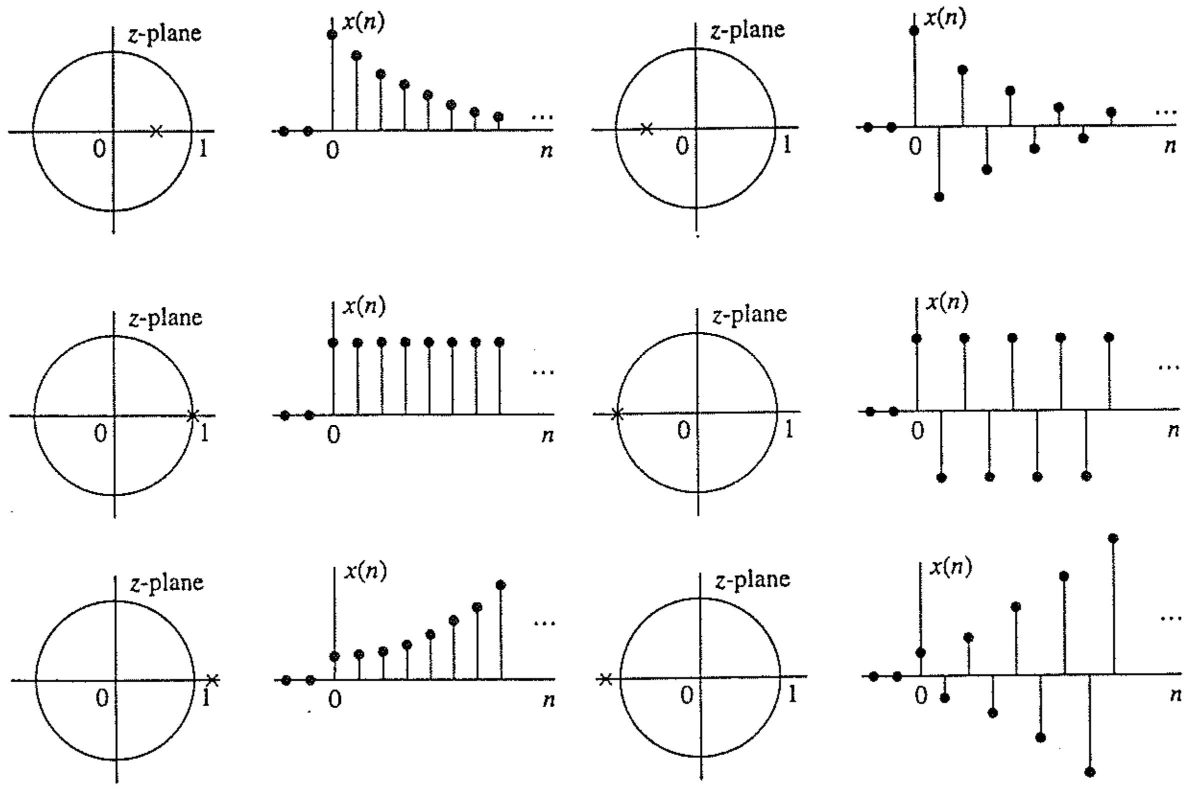
"spike" in $h[n]$

Ex. Single Real Pole

$$\frac{z}{z-a} = \frac{1}{1-az^{-1}} \overset{Z}{\longleftrightarrow} a^n u[n]$$

pole @ $z=a$

ROC $|z|>|a|$



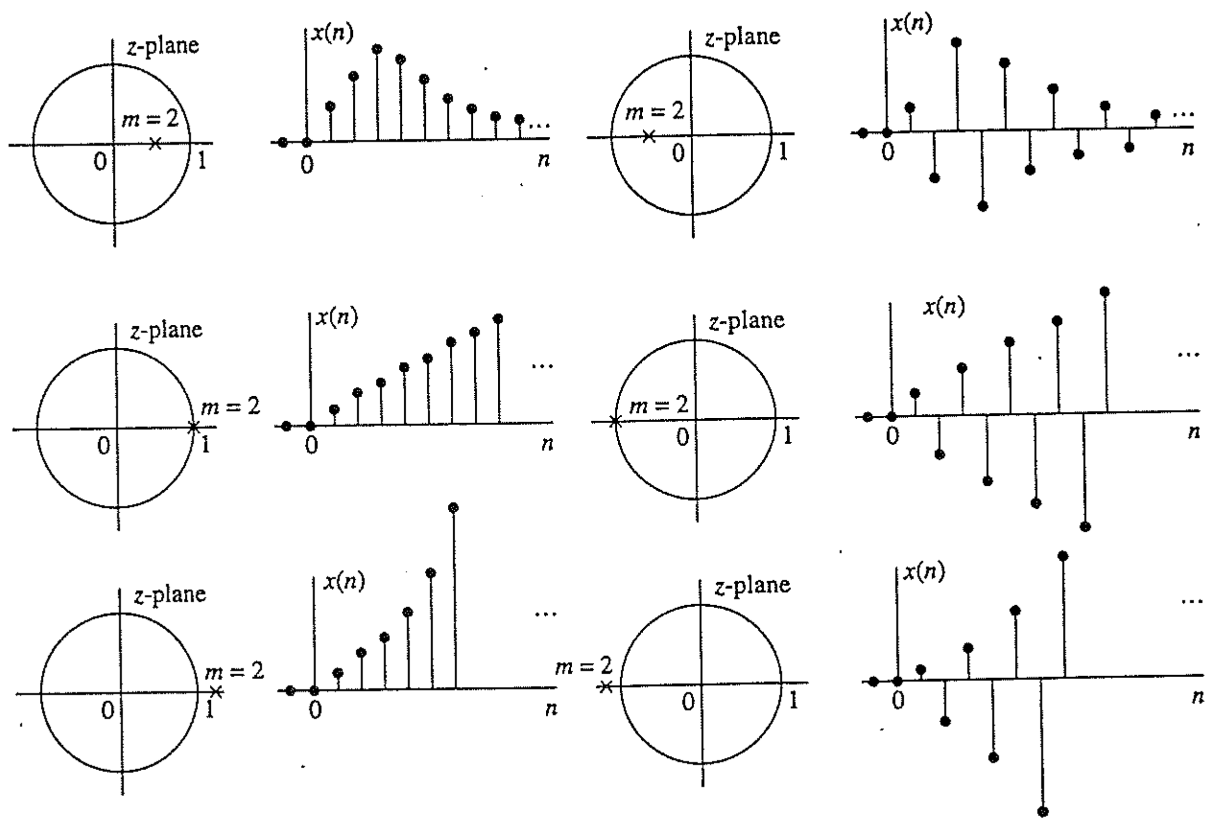
Figures on

from J. Proakis & D. Manolakis "DSP", Prentice-Hall, 1995.

Ex. Double Real Pole

$$\frac{az^{-1}}{(1-az^{-1})^2} \xleftrightarrow{Z} na^n u[n]$$

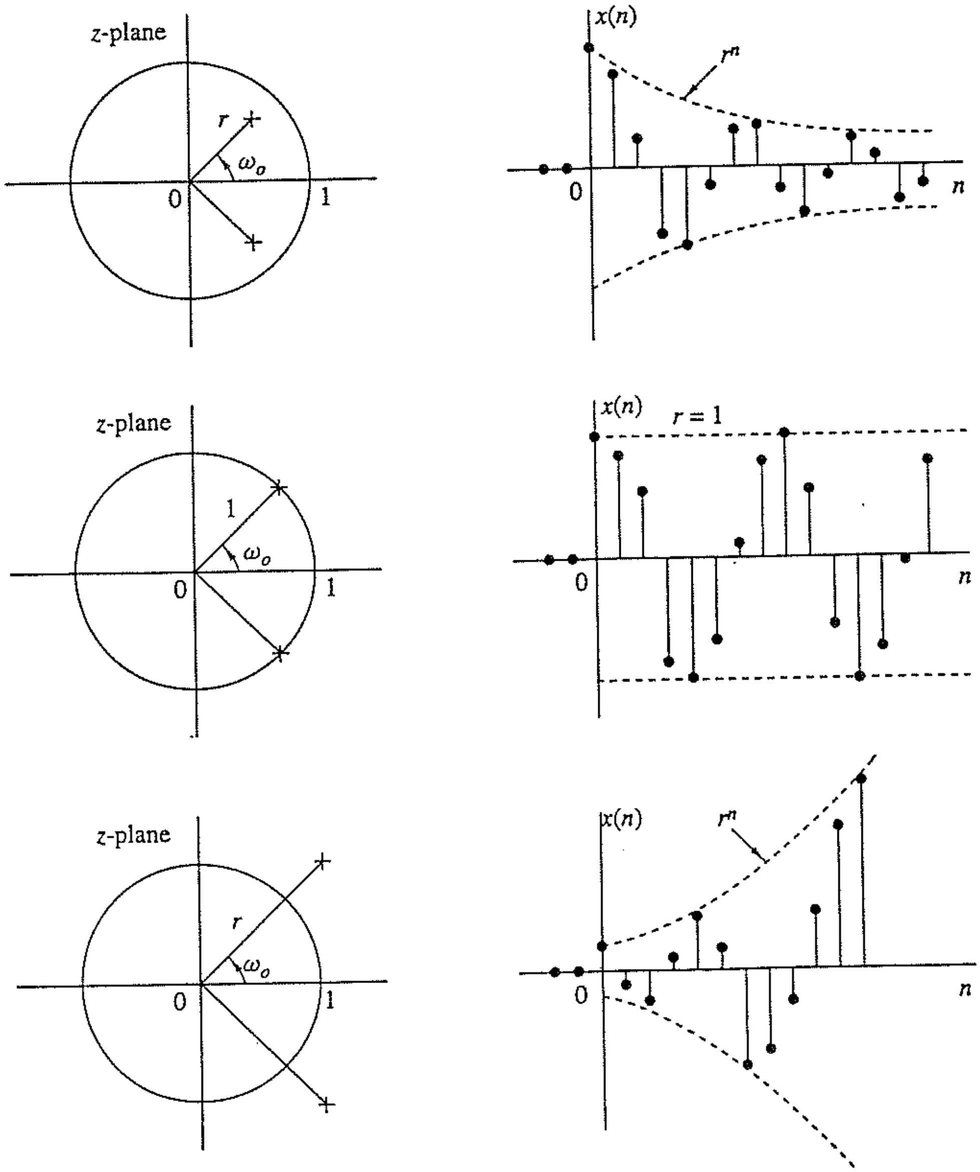
ROC $|z|>|a|$



Ex. Complex Pole Pairs

$$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \overset{Z}{\longleftrightarrow} (\cos\omega_0 n)\, r^n\, u[n]$$

poles @ $z = re^{\pm j\omega_0}$



Frequency Responses

How do poles and zeros influence the frequency response?

We will focus on the magnitude response. See OSB chapter 5 for examination of p

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Recall, BIBO stability condition $\left(\sum_{n=-\infty}^{\infty} h[n] < \infty\right)$ implies that ROC of $H(z)$ includes the unit circle. Therefore, the frequency response (DTFT) of $h[n]$ is well-defined, and is given by

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

We are going to look at

$$|H(\omega)|$$

the <u>magnitude</u> of the freq. response.

By factoring into poles and zeros, we have

$$H(\omega) = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=0}^{N} (1 - d_k e^{-j\omega})}$$

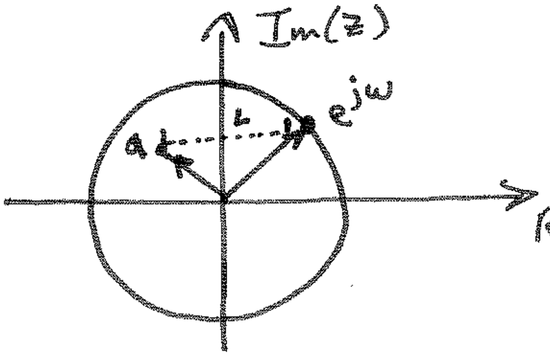
and thus

$$|H(\omega)| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=0}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=0}^{N} |1 - d_k e^{-j\omega}|}$$

Next, multiply by $\frac{|e^{j\omega}|^M}{|e^{j\omega}|^N} = 1$ to get

$$\boxed{H(\omega) = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=0}^{M} |e^{j\omega} - c_k|}{\prod_{k=0}^{N} |e^{j\omega} - d_k|}}$$

What is $|e^{j\omega} - a|$?



$L = |e^{j\omega} - a|$ = length of vector connecting $e^{j\omega}$ and a

$$|H(\omega)| = \left|\frac{b_0}{a_0}\right| \cdot \frac{\prod_{k=0}^{M} |e^{j\omega} - c_k|}{\prod_{k=0}^{N} |e^{j\omega} - d_k|}$$

- c_k - zero $\Rightarrow$ $|e^{j\omega} - c_k|$ = distance from $e^{j\omega}$ to c_k

  $\Rightarrow$ numerator consists of the product of the distances from $e^{j\omega}$ to all <u>zeros</u> of $H(z)$

- d_k - pole $\Rightarrow$ $|e^{j\omega} - d_k|$ = distance from $e^{j\omega}$ to d_k

  $\Rightarrow$ denominator of $|H(\omega)|$ consists of product of the distances from $e^{j\omega}$ to all <u>poles</u> of $H(z)$.

- overall magnitude of frequency response @ ω

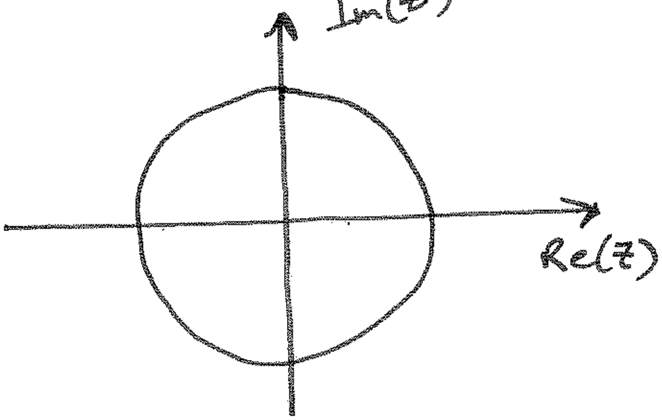
$$|H(\omega)| = \frac{\prod \text{"distances from zeros"}}{\prod \text{"distances from poles"}} \cdot \left|\frac{b_0}{a_0}\right|$$

Therefore,

Close to a zero $\Rightarrow$ magnitude is ________

Close to a pole $\Rightarrow$ magnitude is ________

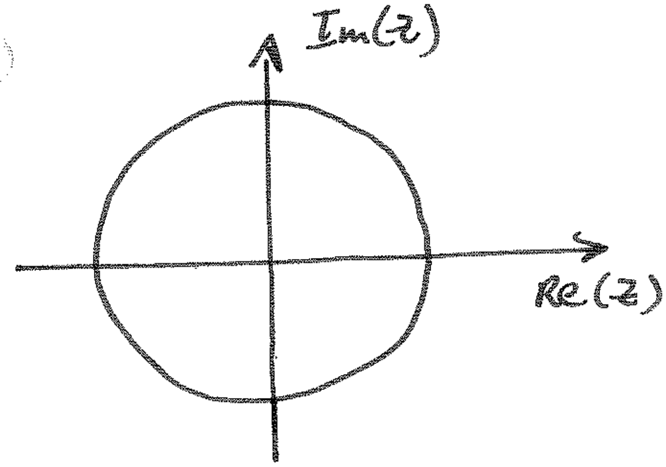
Ex. $H(z) = z^{-1}$, ROC $|z| > 0$, pole @ 0
zero @ $\pm\infty$

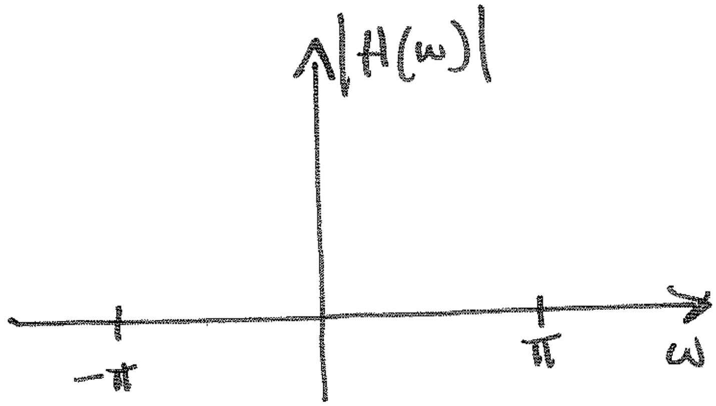


$|H(\omega)| =$

Ex. $H(z) = \frac{z}{z - \frac{1}{2}} = \frac{1}{1 - z^{-1}\frac{1}{2}}$, ROC $|z| > \frac{1}{2}$

zero : $z = 0$
pole : $z = \frac{1}{2}$





$|H(\omega)|^2 =$

Ex. Plot $|H(\omega)|$ corresponding to the pole-zero plot