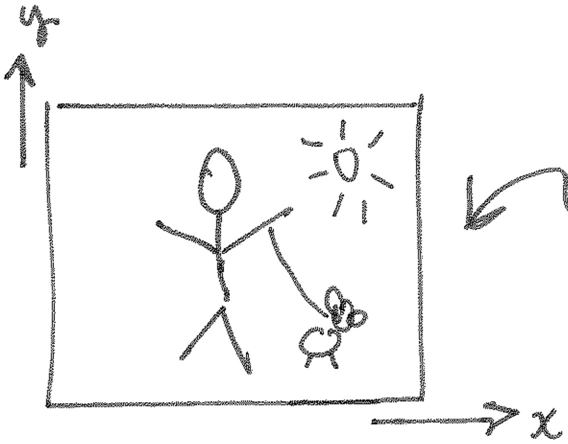
Image Processing

Image Processing



images are 2d functions

$f(x,y)$

Linear Shift Invariant Systems

$$f(x,y) \longrightarrow \boxed{\quad H \quad} \longrightarrow g(x,y)$$

H is LSI if

(1) $$H\Big(\alpha_1 f_1(x,y) + \alpha_2 f_2(x,y)\Big)$$

$$= {}_1 H\big(f_1(x,y)\big) + {}_2 H\big(f_2(x,y)\big)$$

for all images f_1, f_2 and scalar

(ii) $H\left(f(x-x_0, y-y_0)\right)$

$$= g(x-x_0, y-y_0)$$

LSI systems are expressed mathematically as 2D convolutions:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, f(\alpha,\beta)\, d\alpha\, d\beta$$

$h(x,y)$ is the 2D impulse response (also called the "point spread function")

2D Fourier Analysis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-jux}\, e^{-jvy}$$

2D FT

freq variables in x (u) and y (v)

2D Complex Exponentials are Eigenfunctions for 2D LSI systems.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, e^{+ju_0\alpha}\, e^{+jv_0\beta}\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha', \beta')\, e^{+ju_0(x-\alpha')}\, e^{+jv_0(y-\beta')}\, d\alpha' d\beta'$$

change of variables
$\alpha' = x-\alpha$, $\beta' = y-\beta$

$$= e^{+ju_0 x} \cdot e^{+jv_0 y} \cdot \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha', \beta')\, e^{-ju_0\alpha'}\, e^{-jv_0\beta'}\, d\alpha' d\beta'}_{\equiv\ H(u_0, v_0)}$$

2D Fourier transform of $h(x,y)$ evaluated at freq (u_0, v_0)

$$f(x,y) \longrightarrow \boxed{\begin{matrix} \text{LSI} \\ H \end{matrix}} \longrightarrow g(x,y)$$

$$g(x,y) = h(x,y) * f(x,y)$$

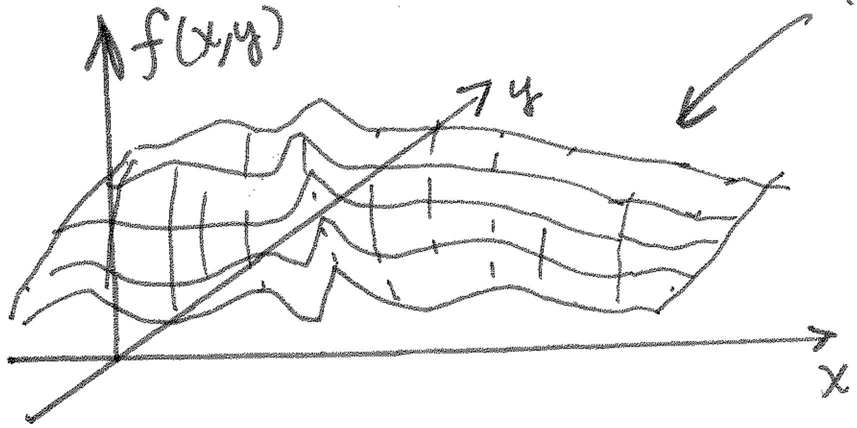
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, f(\alpha,\beta)\, d\alpha\, d\beta$$

$$G(u,v) = H(u,v) \cdot F(u,v)$$

$$g(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)\, e^{jux}\, e^{jvy}\, du\, dv$$

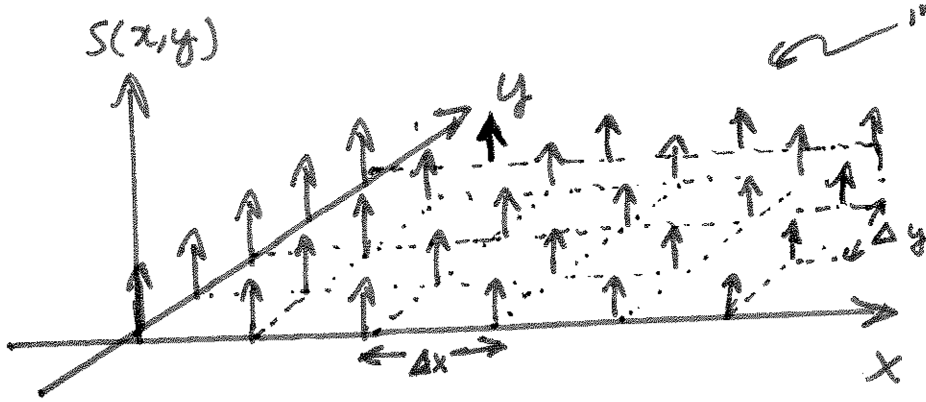
inverse 2D FT

2D Sampling Theory



think of the image as a 2D surface

We can "sample" the height of the surface using a 2D impulse array.



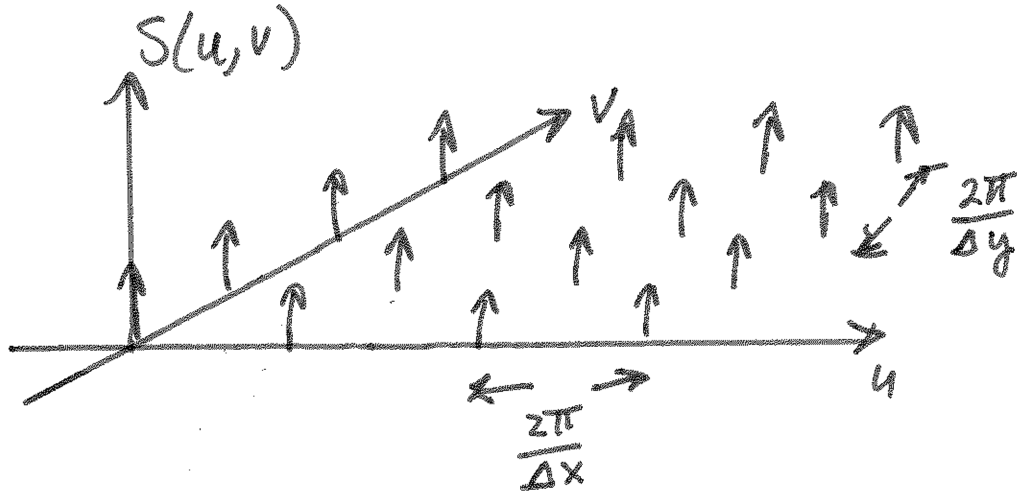
impulses spaced Δx apart in horizontal direction and Δy in vertical

$$f_s(x,y) = s(x,y) \cdot f(x,y)$$

sampled image

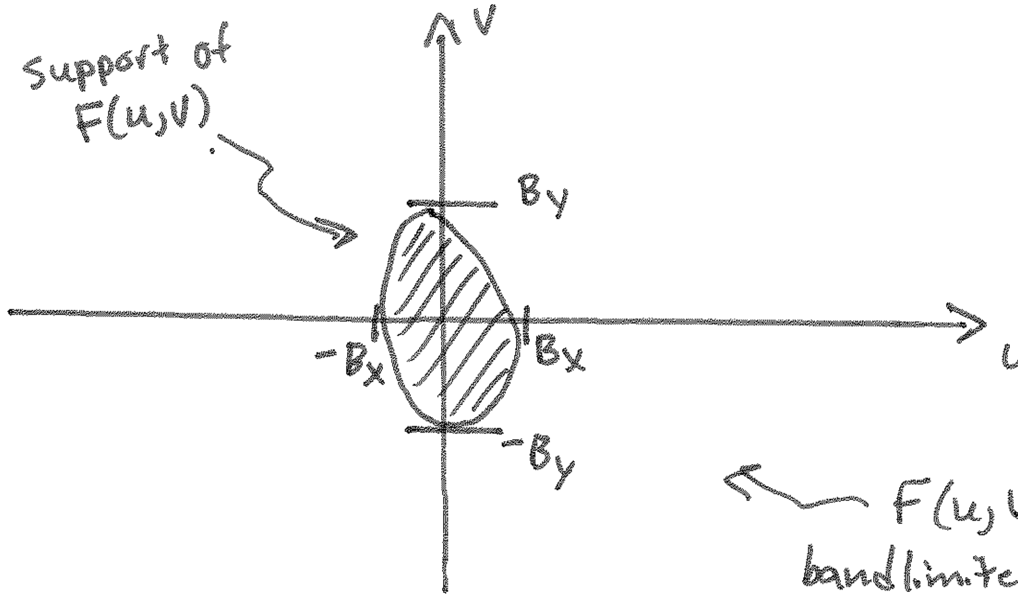
... in frequency

2D FT of $s(\ y)$ is a 2D impulse array in frequency $S(u,v)$



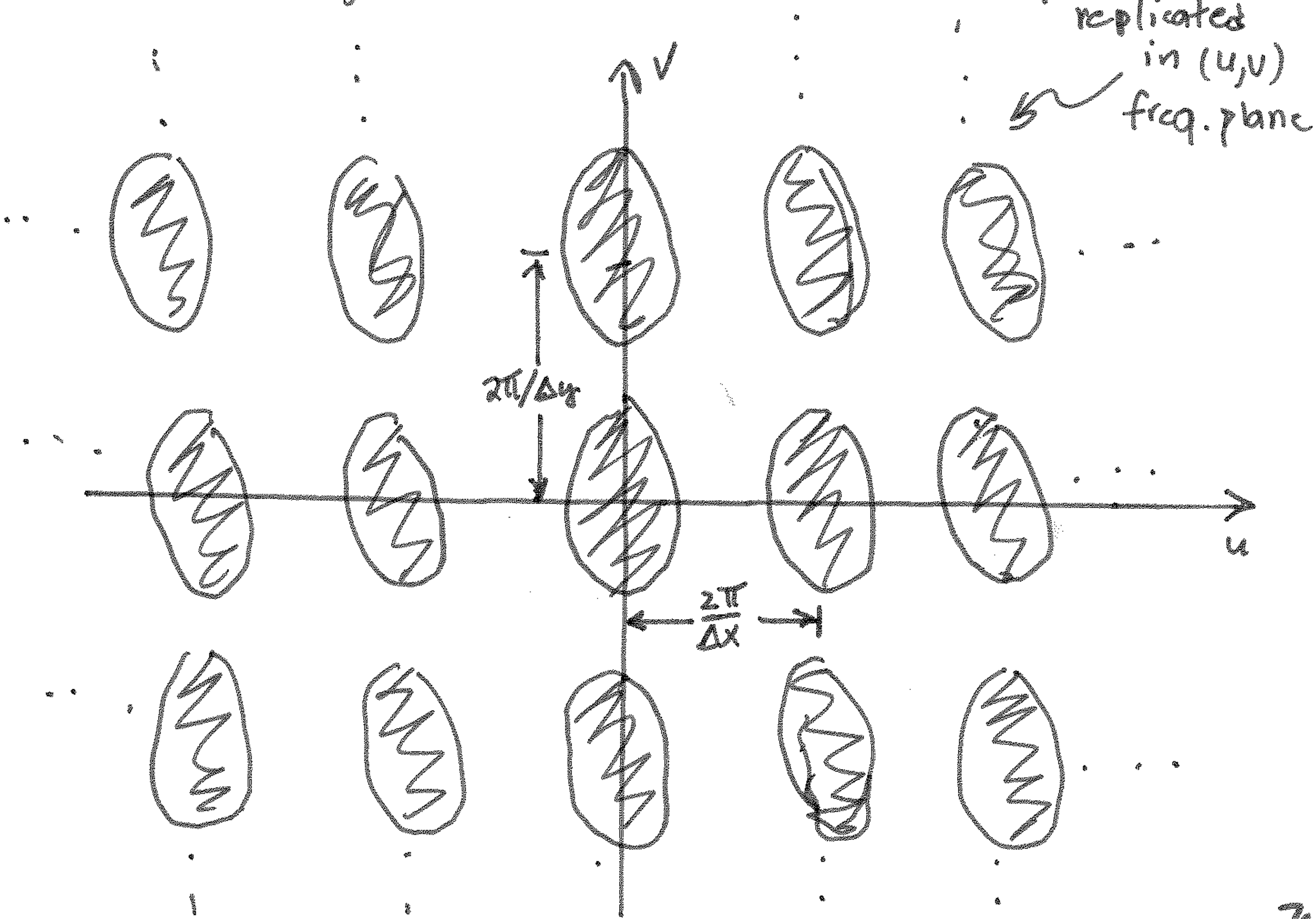
mult. in time $\Longleftrightarrow$ convolution in freq.

$$F_s(u,v) = S(u,v) * F(u,v)$$



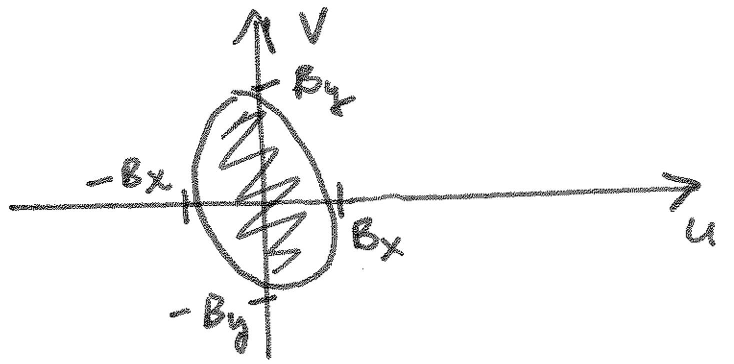
F(u,v) is bandlimited in horizontal & vertical directions

$F_s(u,v)$



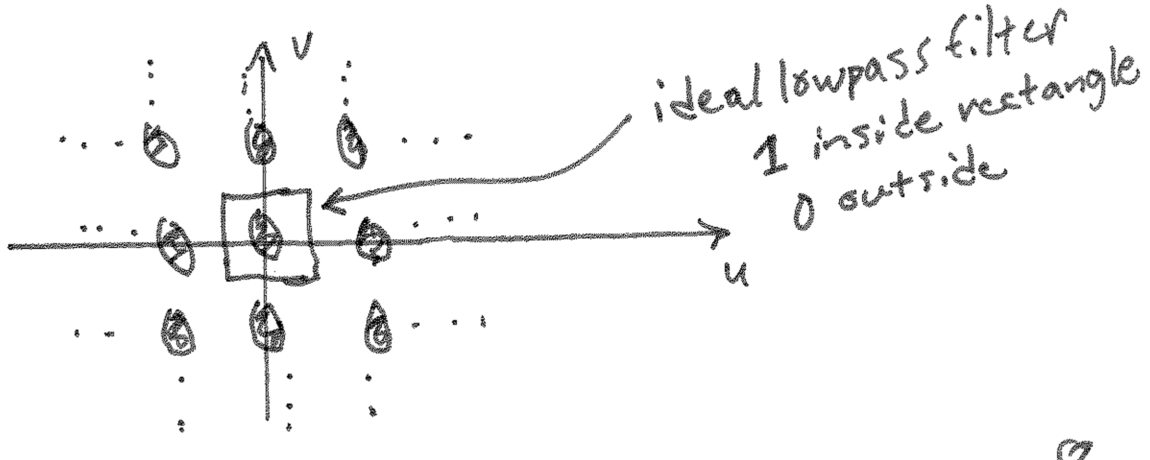
Nyquist Theorem.

Assume that $f(x,y)$ is bandlimited to $\pm B_x$, $\pm B_y$:

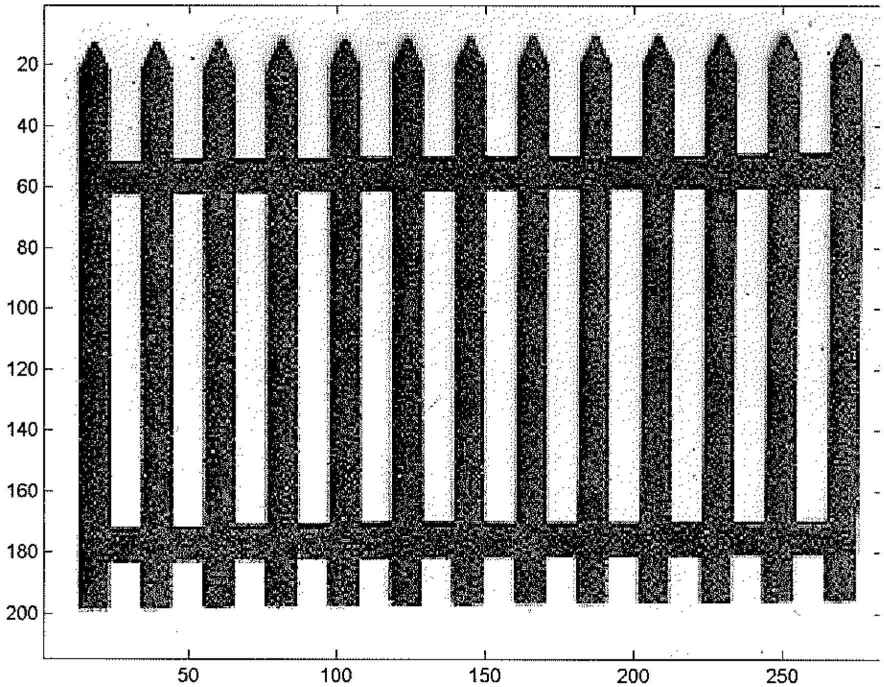


If we sample $f(x,y)$ at spacings of $\Delta x < \frac{\pi}{B_x}$ and $\Delta y < \frac{\pi}{B_y}$,

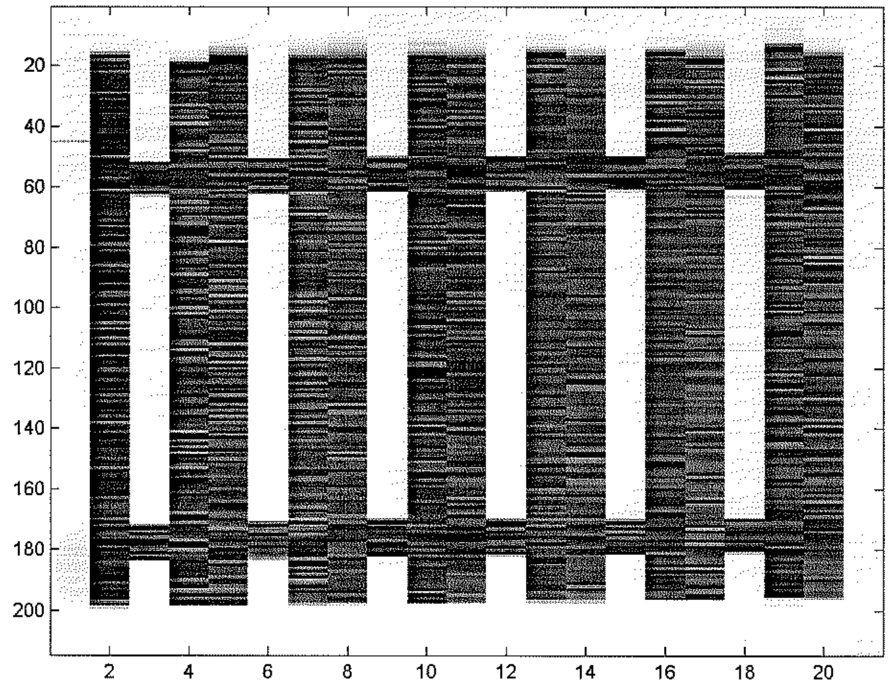
then $f(x,y)$ can be perfectly recovered from the samples by lowpass filtering:



Aliasing in 2D



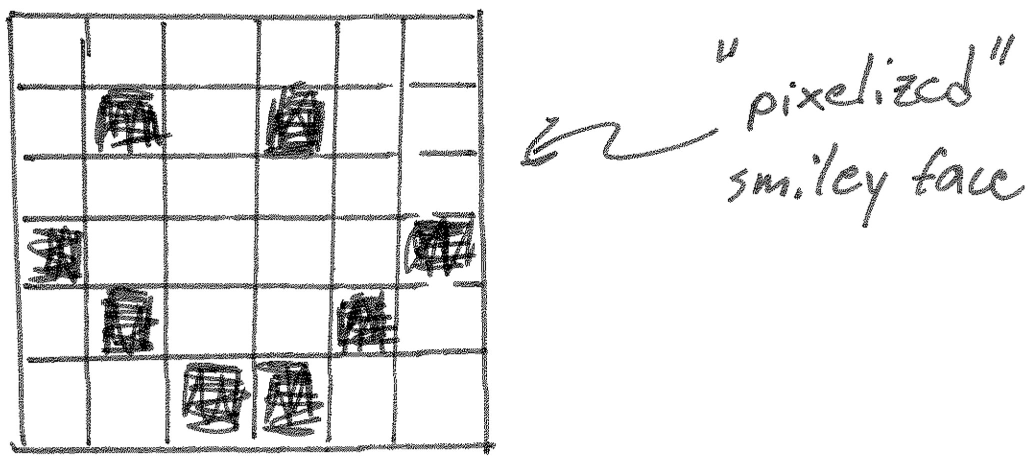
20 samples/cycle



1.4 samples/cycle

Digital Image Processing

A sampled image gives us our usual 2D array of pixels $f[m,n]$



We can filter $f(m,n)$ by applying a 2D discrete-space convolution

$$g[m,n] = h[m,n] * f[m,n]$$

PSF

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h[m-k, n-l]\, f[k,l]$$

Sampled Image

close-up shows "pixelized" (sampled)
nature of the image

We also have discrete-space FTs:

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-jum} e^{-jvm}$$

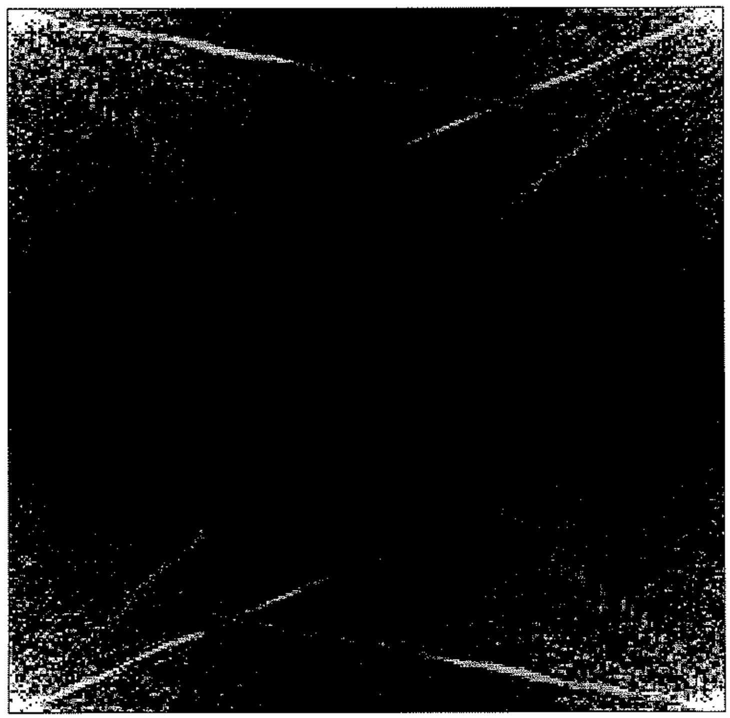
analogous to DTFT in 1D

Convolution in time $\Longleftrightarrow$ multiplication in frequency

$$g[m,n] = h[m,n] * f[m,n]$$

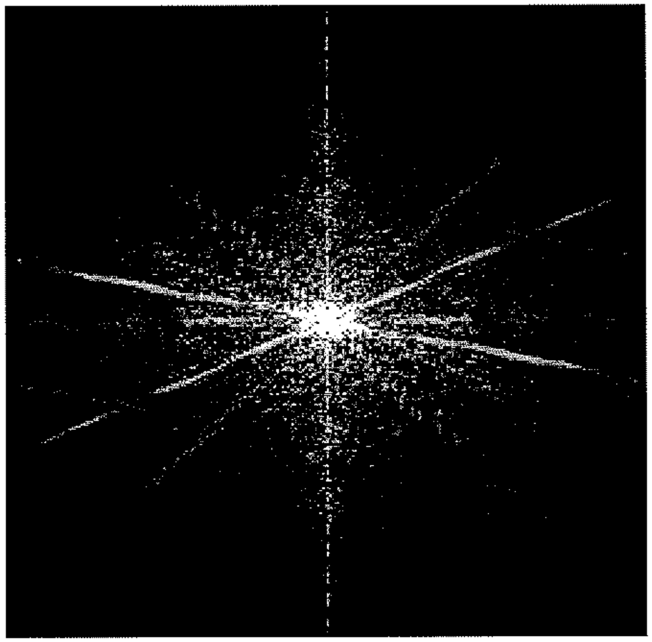
$$\Longleftrightarrow \quad G(u,v) = H(u,v) \cdot F(u,v)$$

Magnitude of FT of Camerman image



Magnitude of FT of Camerman image

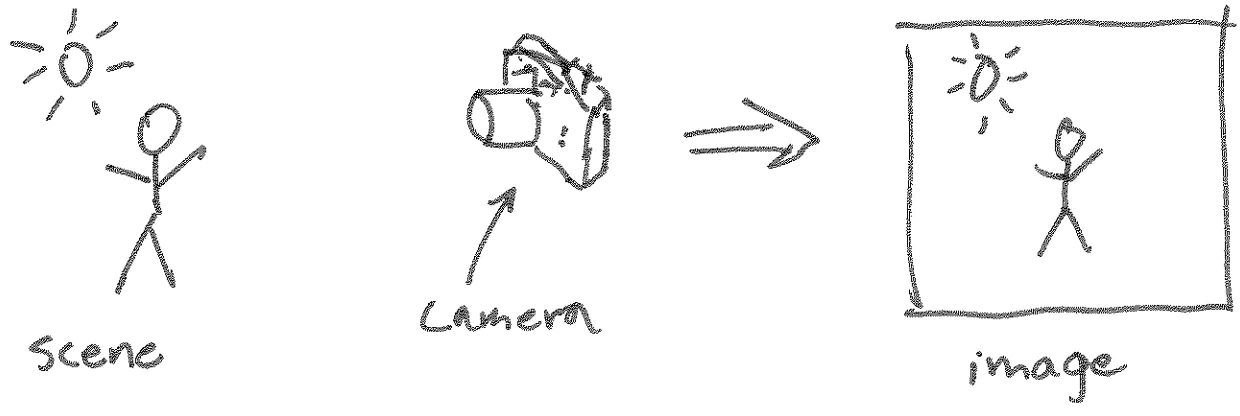
$-\pi, -\pi$ $\qquad\qquad$ $\pi, -\pi$



$\pi, -\pi$ $\qquad\qquad$ π, π

fftshift command in Matlab centers FT at 0,0
(DC is now in the middle of the image)

Application : Image Restoration



In many applications (e.g., satellite imaging, medical imaging, astronomical imaging, poor quality family portraits) the imaging system introduces a slight distortion.

Often, images are slightly blurred and image restoration aims at <u>deblurring</u> the image.

The blurring can usually be modeled as an LSI system with a given PSF $h[m,n]$.

$$h[m,n] \xleftrightarrow{FT} H(u,v)$$

The observed image is

$$g[m,n] = h[m,n] * f[m,n]$$

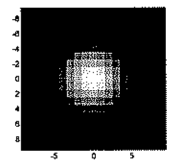
$$G(u,v) = H(u,v) \cdot F(u,v)$$

$$F(u,v) =$$

Image Blurring

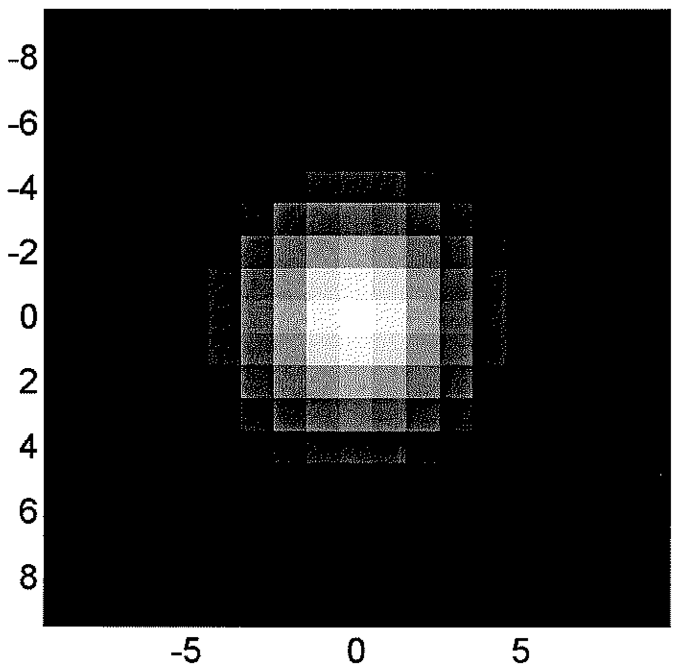
image

*

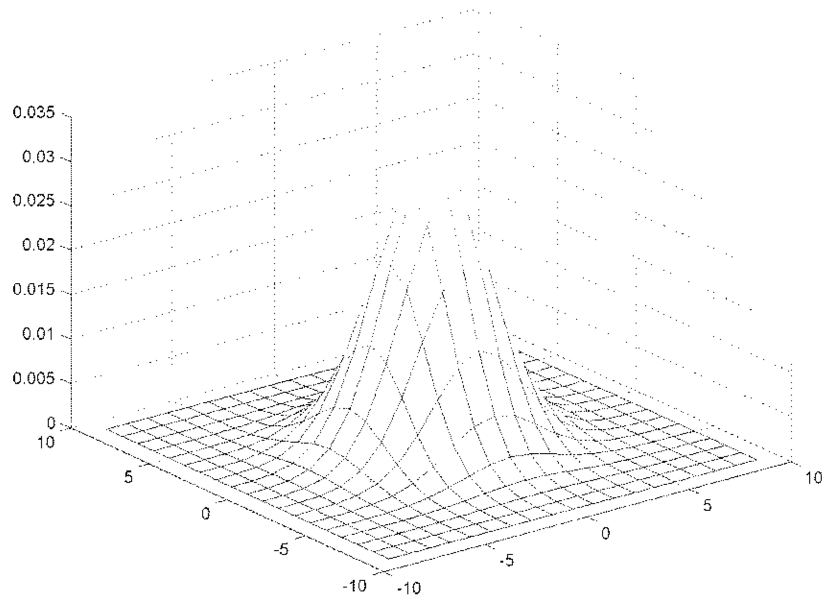


blur

Image Blurring



blur



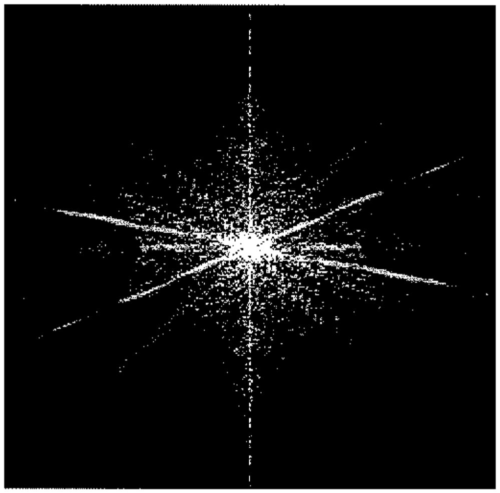
surface plot

Image Blurring

original

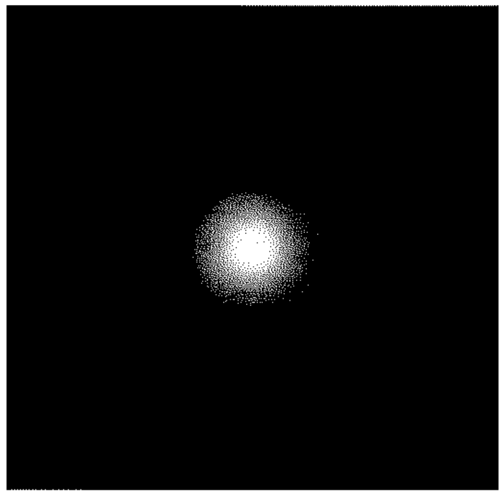
blurred

Image Blurring (frequency domain)



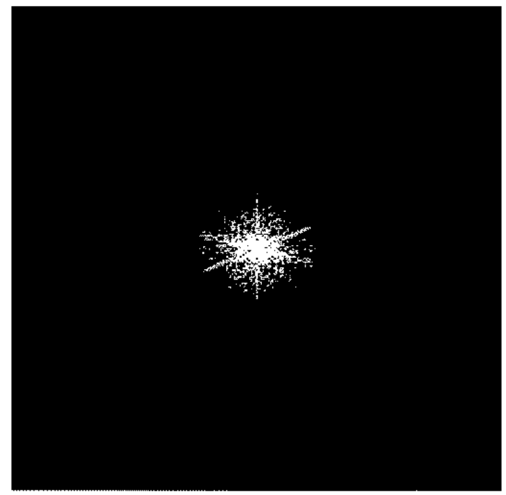
FT of original camerman

×



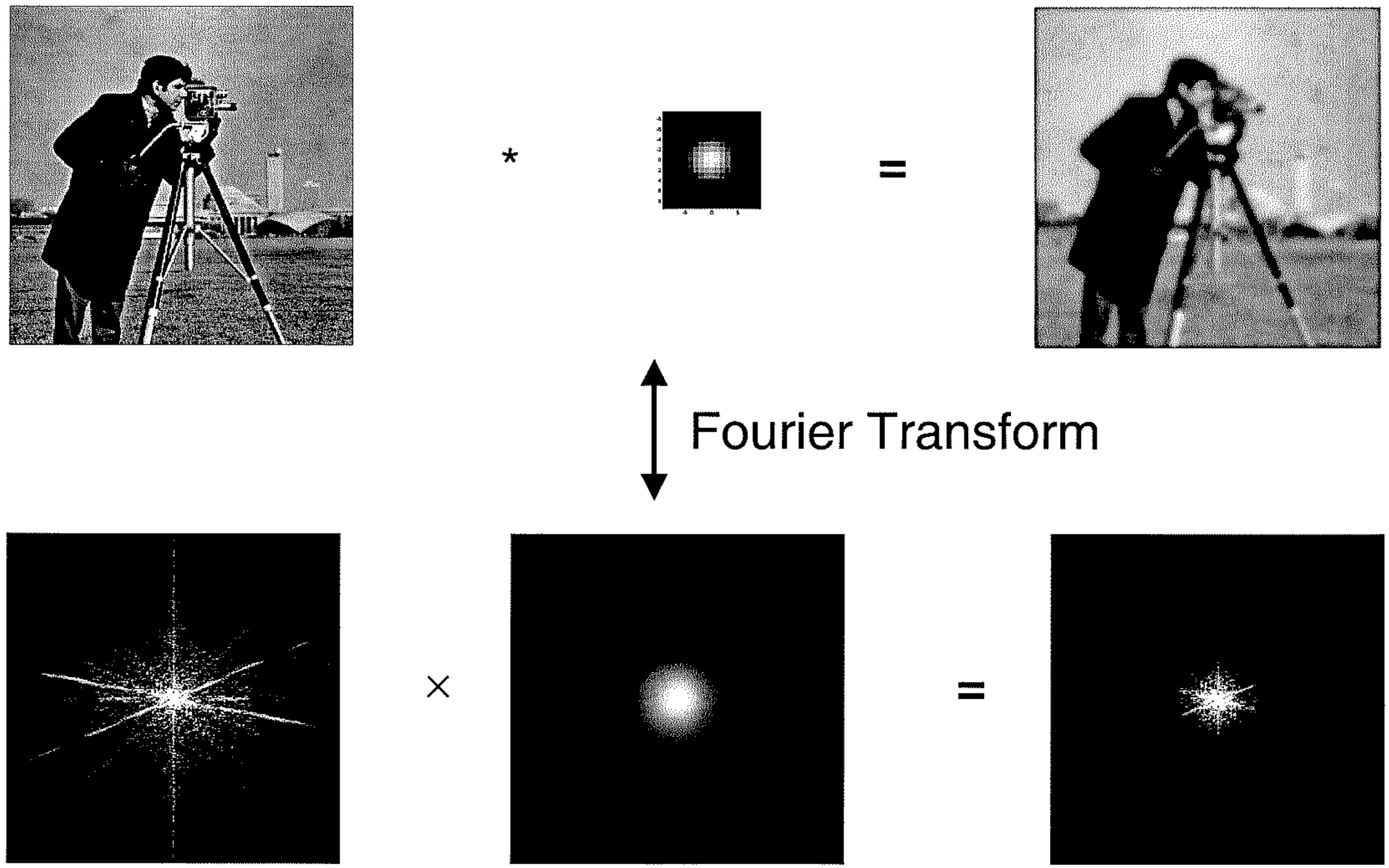
FT of blur point spread function

=



FT of blurred image

Image Blurring



2D DFT

To perform image restoration (and many other useful image processing algorithms) in a computer, we need a FT that is discrete.

$$F[k,\ell] = F(u,v)\Big|_{u=\frac{2\pi}{N}k,\ v=\frac{2\pi}{N}\ell}$$

$k = 0, \ldots, N-1$
$\ell = 0, \ldots, N-1$

2D DFT

$$F(u,v) = \sum_{m=} \sum_{n=} f[m,n]\, e^{-jum} e^{-jvm}$$

$$F[k,l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m,n]\, e^{-j\frac{2\pi}{N} m} e^{-j\frac{2\pi}{N} ln}$$

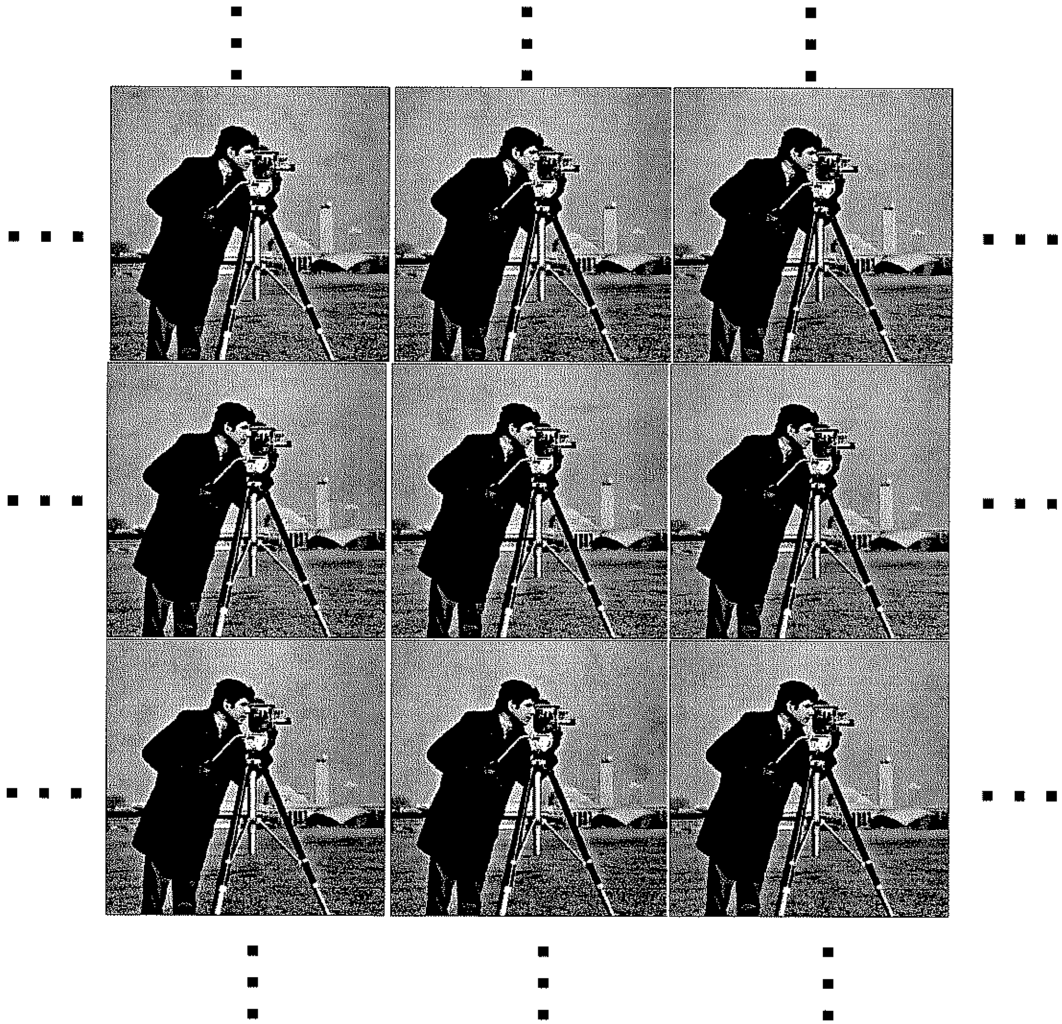
finite support
$N \times N$ image

Inv. 2D DFT

$$f[m,n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F[k,l]\, e^{j\frac{2\pi}{N} km} e^{j\frac{2\pi}{N} ln}$$

reconstruct image
as a weighted combo
of complex sinusoidal
basis functions

Periodic extension



2D DFT and Convolution

Regular 2D convolution:

$$g[m,n] = \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} h[m-k, n-\ell]\, f[k,\ell]$$

flipped horz & vert.

ex.

$$h = \begin{bmatrix} h[0,0] & h[0,1] \\ h[1,0] & h[1,1] \end{bmatrix} \qquad h[m,n] = 0 \text{ for } \begin{array}{l} m,n > 1 \\ m,n < 0 \end{array}$$

$$f = \begin{bmatrix} f[0,0] & \cdots & f[0,N-1] \\ \vdots & \ddots & \vdots \\ f[N-1,0] & \cdots & f[N-1,N-1] \end{bmatrix}$$

Flip h

$$
\begin{array}{c c}
h[-m,-n] = \begin{bmatrix} h[1,1] & h[1,0] & 0 \\ h[0,1] & h[0,0] & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} n = \\ -1 \\ 0 \\ 1 \end{array} \\
\begin{array}{cccc} m = & -1 & 0 & 1 \end{array} &
\end{array}
$$

Convolve

$$
g[0,0] = \begin{bmatrix} h[1,1] & h[1,0] \\ h[0,1] & h[0,0] \end{bmatrix} \begin{bmatrix} \times f[0,0] & f[0,1] & \cdots \\ [1,0] & \ddots & \\ \vdots & & \end{bmatrix}
$$

$$
= h[0,0] \times f[0,0]
$$

$$
g[0,1] = \begin{bmatrix} h[1,1] & h[1,0] & & \\ h[0,1] \times f[0,0] & h[0,0] \times f[0,1] & f[0,2] & \cdots \\ f[1,0] & \ddots & & \\ \vdots & & \ddots & \\ & & & \end{bmatrix}
$$

$$
= h[0,0] \times f[0,1] + h[0,1] \times f[0,0]
$$

$$
g[m,n] = h[0,0] \times f[m,n] + h[0,1] \times f[m,n-1] + h[1,0] \times f[m-1,n] + h[1,1] \times f[m-1,n-1]
$$

What does $H[k,l] \cdot F[k,l]$ produce?

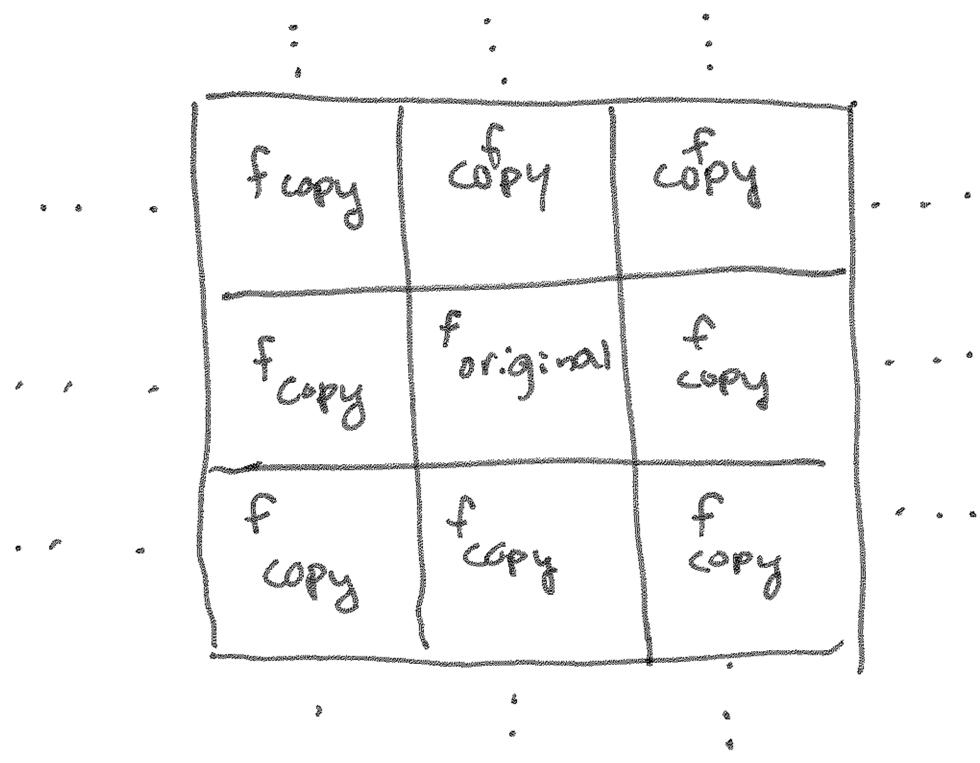
DFTs of h, f

Answer: 2D circular convolution

$$\tilde{g}[m,n] = \text{IDFT}\Big(H[k,l] \cdot F[k,l] \Big)$$

$= \text{circular conv in 2D}$

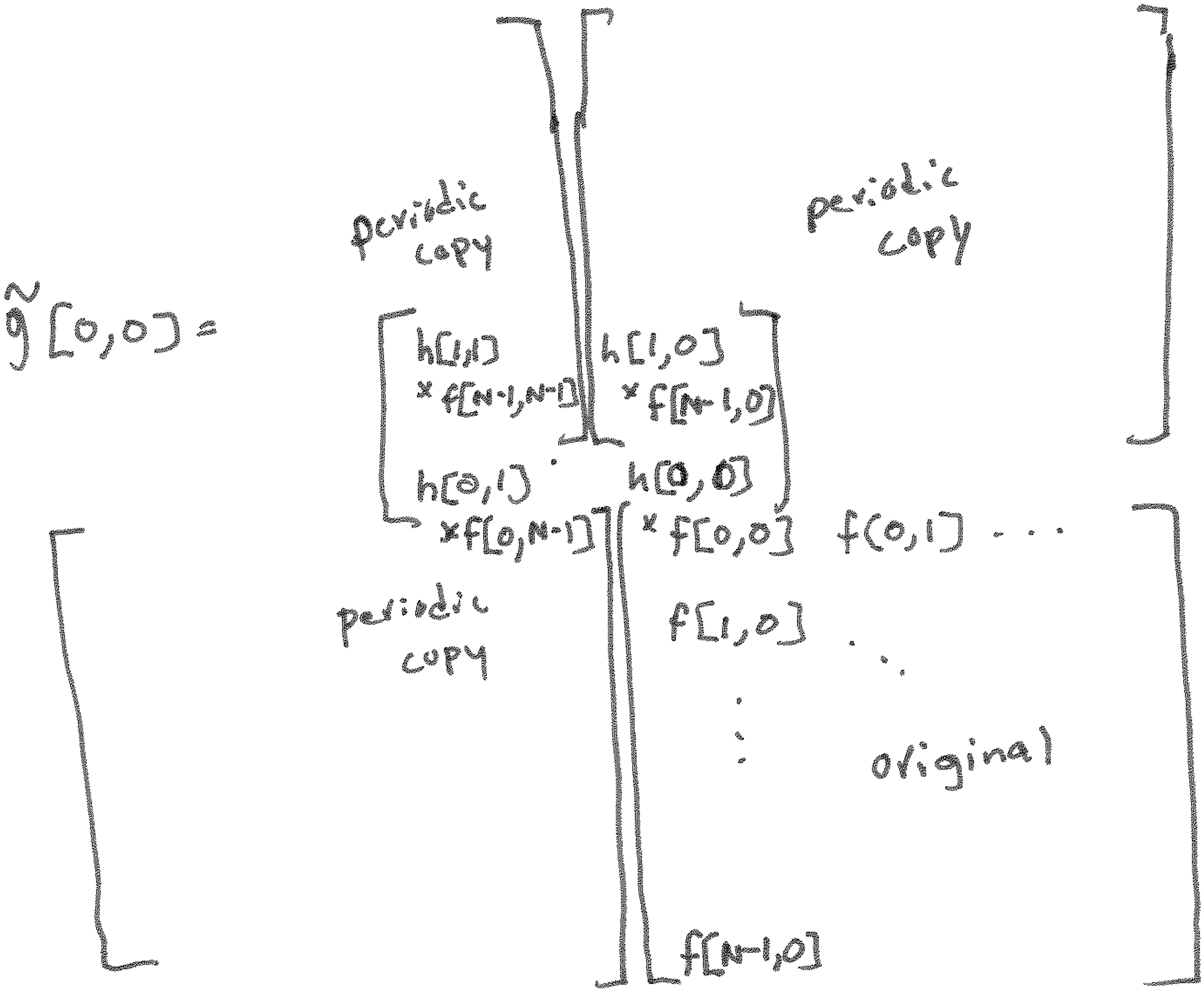
Due to periodic extension by DFT:



Let

$$\tilde{g}[m,n] = \mathrm{IDFT}\Big(H[k,\ell] \cdot F[k,\ell] \Big)$$

$\tilde{g}[0,0] =$

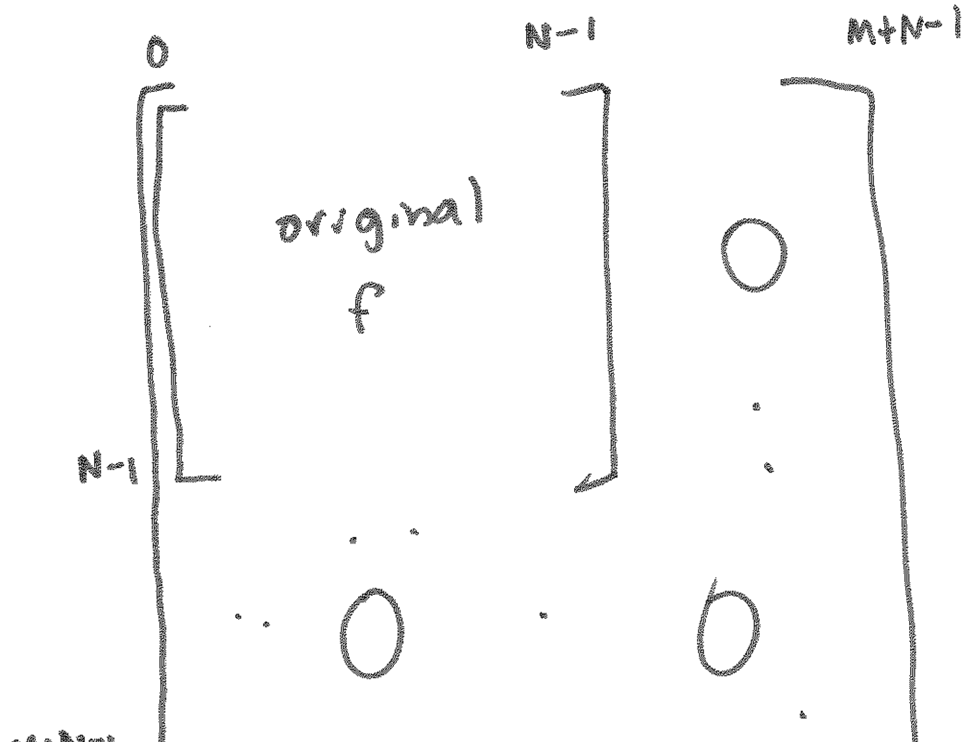


$$\tilde{g}[0,0] = h[0,0]\,f[0,0] + h[1,0]\,f[N-1,0] + \underbrace{h[0,1]\,f[0,N-1] + h[1,1]\,f[N-1,N-1]}$$

wrap around effect

Zero Padding

If the support of h is $M \times M$ and f is $N \times N$, the zeropad f and h to $M+N\ 1 \times M+N-1$



$\Longrightarrow$ circular conv = regular conv.

Computing the 2D DFT

$$F[k,l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m,n] \, e^{-j\frac{2\pi}{N}km} \, e^{-j\frac{2\pi}{N}ln}$$

$$= \sum_{m=0}^{N-1} \underbrace{\left(\sum_{n=0}^{N-1} f[m,n] \, e^{-j\frac{2\pi}{N}ln} \right)}_{\substack{\text{1D DFT over } n \\ \text{- take 1D FFT of each row}}} e^{-j\frac{2\pi}{N}km}$$

1D DFT over n
- take 1D FFT of each row

} N rows, N log N operations per row

$$= \sum_{m=0}^{N-1} f'[m,l] \, e^{-j\frac{2\pi}{N}km}$$

1D FFT of each column } N columns, N log N ops per column

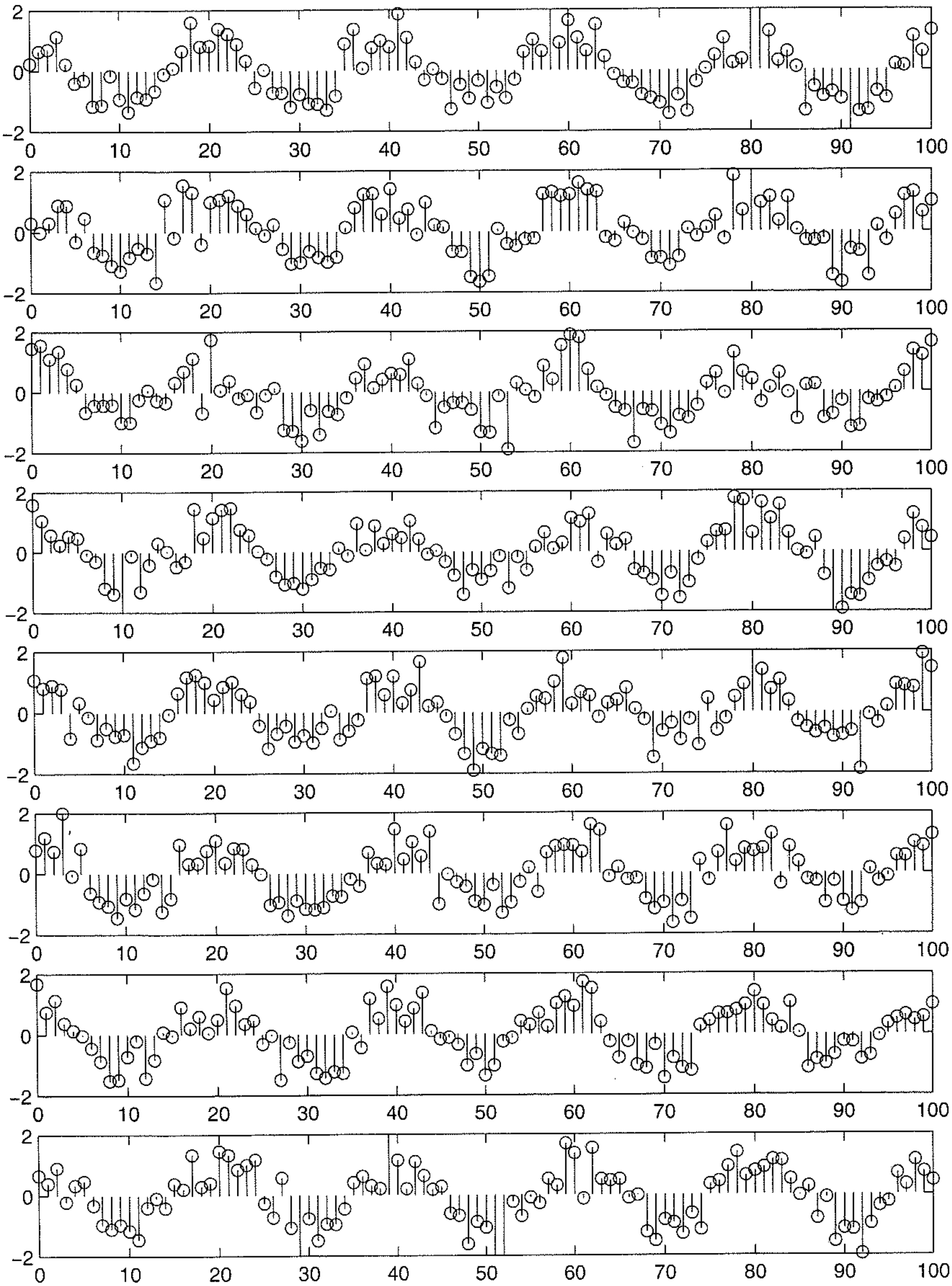
Overall complexity of 2D FFT

$$O(N^2 \log N)$$

N^2 = # of pixels

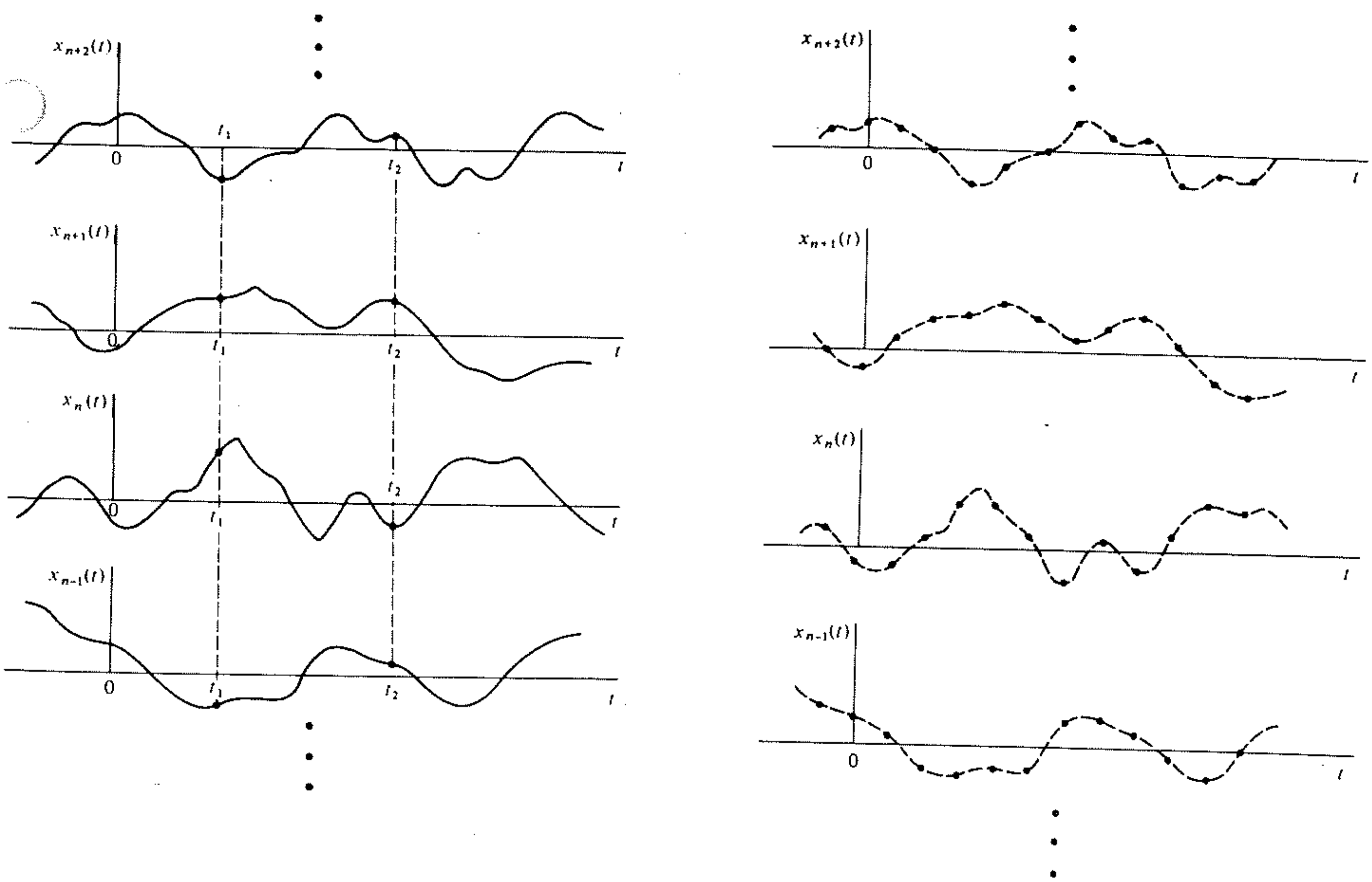
Discrete-Time Random Signals

Classic Example: Sinusoid + Noise

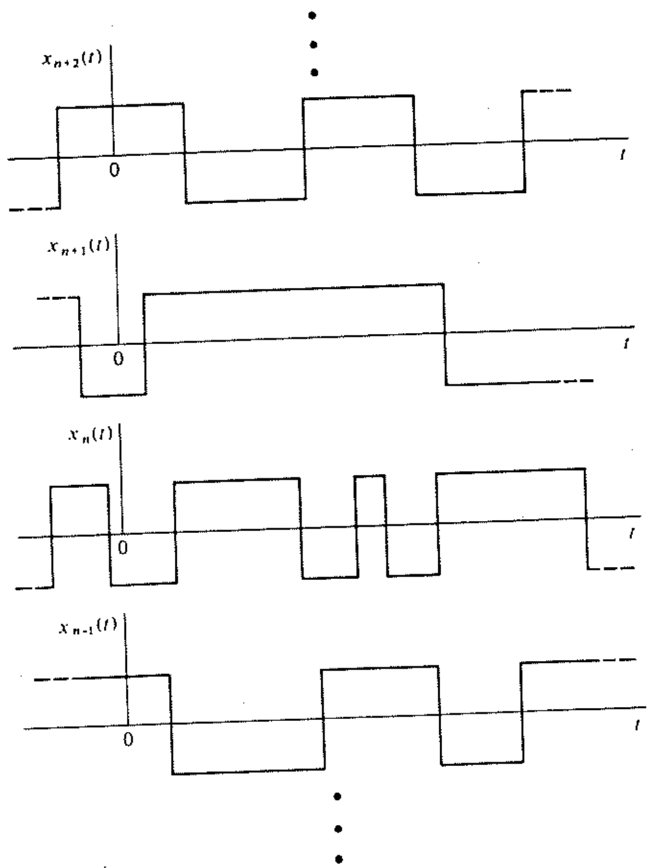


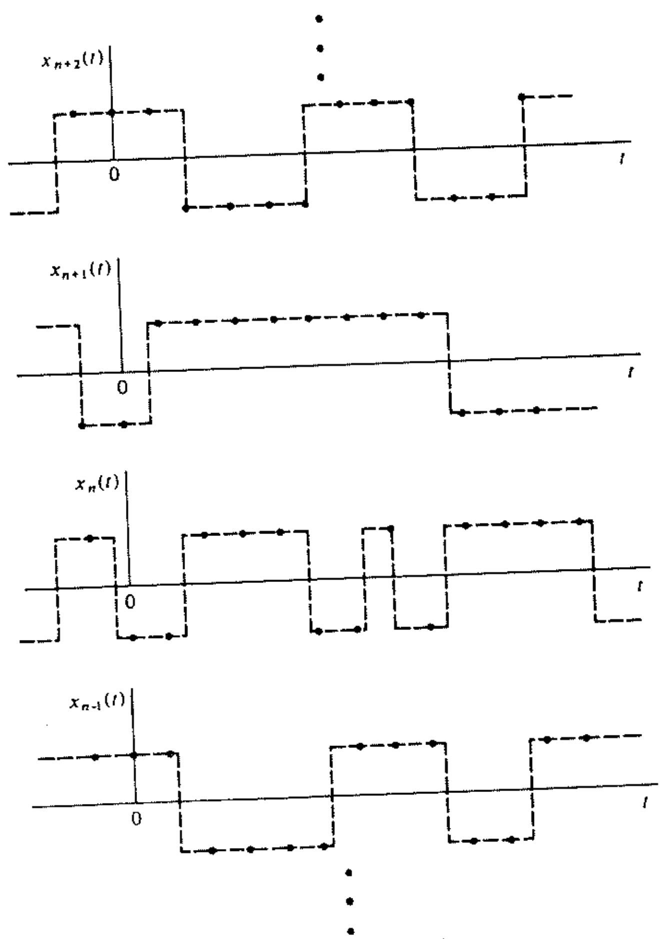
Definition : A random process is a family or ensemble of signals corresponding to every possible outcome of a certain signal measurement or experiment. Each signal in the ensemble is called a "realization" of the process.

Ex.



Ex. Random Binary Process



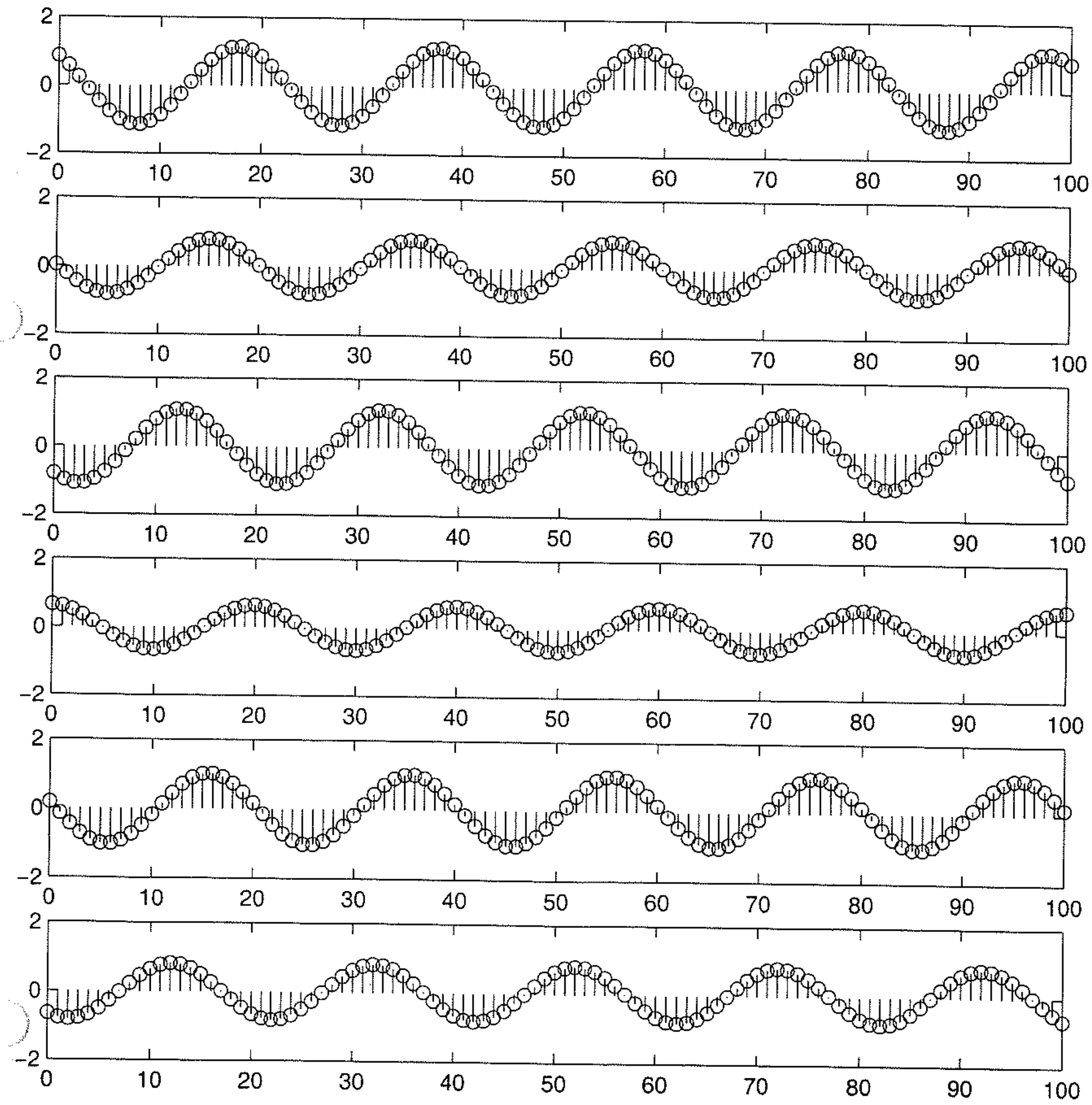


Ex. Random Sinusoidal Process

$$x[n] = A\cos(\omega_0 n + \phi)$$

A, ω_0, ϕ may all be random variables

random amplitude & phase



Definition: The *mean* of a random process is the average of all realizations of the process.

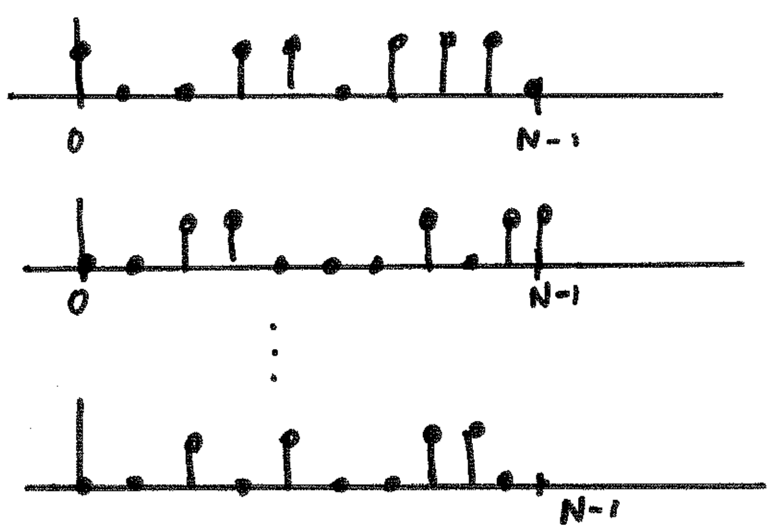
Ex. N-pt binary random sequence

$$x[n] = \begin{cases} 1, & \text{with prob } p \\ 0, & \text{with prob } (1-p) \end{cases} \qquad n = 0, \ldots, N-1$$

independent

$$\Pr(x[n]=1) = p \qquad \Pr(x[n]=0) = 1-p$$

Realizations:



What is mean$(x[n])$?

The mean is also known as the expectation, and is denoted by

$$m_x[n] = E\left[x[n]\right]$$

expectation operator

$\Rightarrow$ take average of all possible realizations of $x[n]$

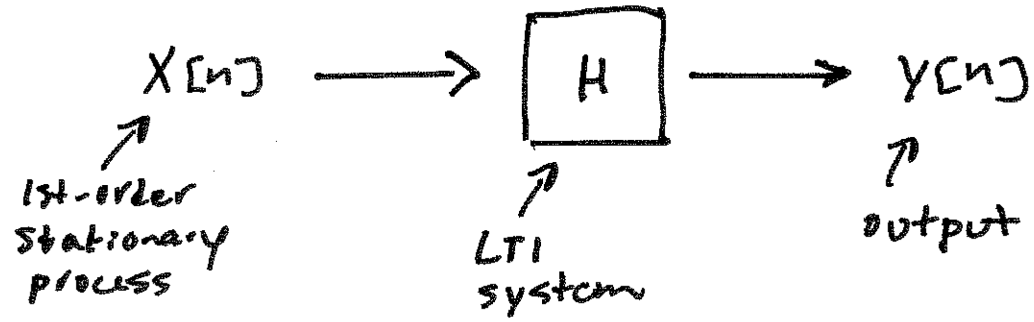
First-order Stationary Processes

A process is 1st-order stationary if

$$m_x[n] = m_x \text{ (a constant independent of } n\text{)}$$

Ex. N-pt binary process revisited

First-order Stationarity and LTI Systems



LTI system is deterministic, but input is random.

What about output?

$$m_Y[n] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right]$$

$$=$$

<u>Definition</u>: The <u>autocorrelation function</u> of a random process is the average product of a signal realization with a time-shifted version of itself.

$$R_{xx}[n, n+m] = E\left[\, x[n]\, x[n+m] \,\right]$$

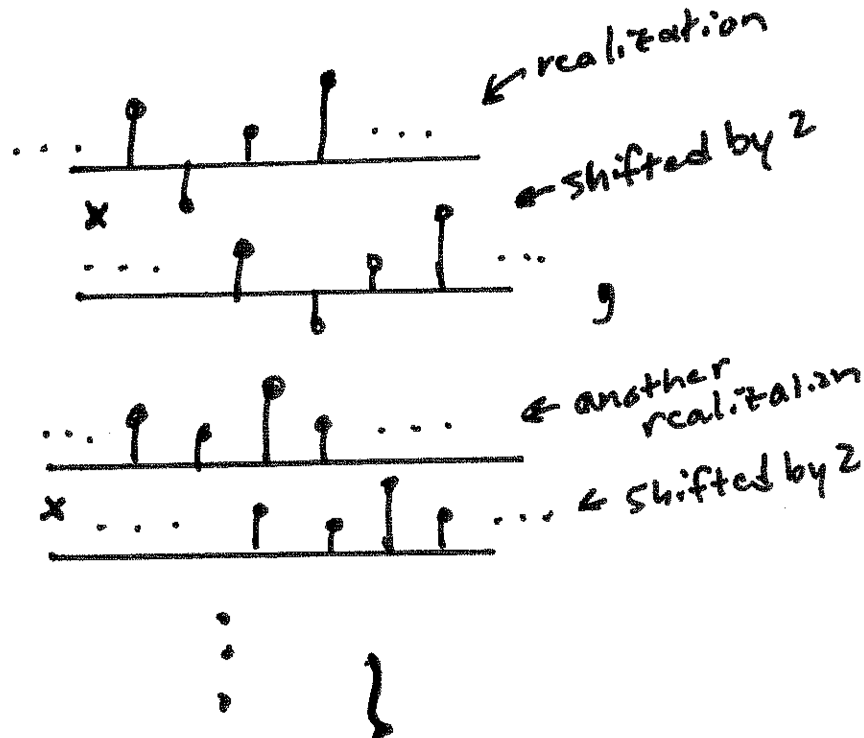
autocorrelation function

Average over all possible realizations

<u>Ex</u>.

$R_{xx}[n, n+2] =$

Average $\Bigg\{$



$\Bigg\}$

Ex. Random Binary Process

Assume $\Pr\left(x[n]=1\right) = \Pr\left(x[n]=-1\right) = \frac{1}{2}$

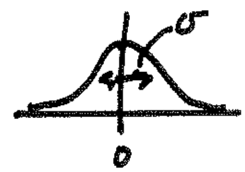
$$\Longrightarrow \quad m_x[n] =$$

$$R_{xx}[n, n+m] = E\left[x[n]\,x[n+m]\right]$$

Ex. Gaussian White Noise (GWN)

"white" $\Rightarrow$ $x[n]$ are independent and zero-mean

$$\Pr\left[a \le x[n] \le b\right] = \int_a^b \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}}_{\substack{\text{Gaussian density}\\ \text{function}\\ \sigma^2 = \text{variance}}} dx$$



$$m_x = E\left[x[n]\right] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}\, dx = 0$$

$$R_{xx}[n, n+m] = E\left[x[n]\, x[n+m]\right]$$

$$=$$

Second-order Stationary Processes

A random process is 2nd-order stationary if

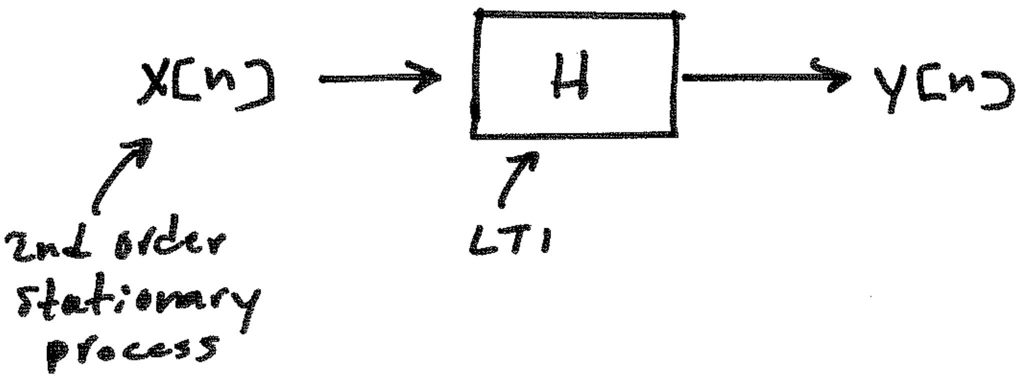
$$R_{xx}[n, n+m] \equiv R_{xx}[m]$$

That is, the autocorrelation function only depends on m, the shift, and not on n.

Ex. Binary Process

Ex. GWN

Stationary Inputs & LTI Systems



$$m_y[n] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, E\left[x[n-k]\right] = m_x \sum_{k=-\infty}^{\infty} h[k]$$

$$R_{yy}[n, n+m] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \cdot \sum_{r=-\infty}^{\infty} h[r]\, x[n+m-r]\right]$$

$$=$$

$$\Rightarrow R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} R_{xx}[m-\ell] \cdot c_{hh}[\ell]$$

where $c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]\, h[\ell+k]$

Ex. GWN into an LTI System

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

GWN

$$m_y = m_x \sum_{k=-\infty}^{\infty} h[k] = 0$$

$$R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \overset{\sigma^2 \delta[m-\ell]}{\cancel{R_{xx}[m-\ell]}} \cdot \sum_{k=-\infty}^{\infty} h[k]\, h[\ell+k]$$

$$= \sum_{\ell=-\infty}^{\infty} \sigma^2 \delta[m-\ell] \cdot \sum_{k=-\infty}^{\infty} h[k]\, h[\ell+k]$$

$$= \sum_{k=-\infty}^{\infty} \sigma^2 h[k]\, h[m+k]$$

Frequency Domain Analysis

(assume 2nd-order stationarity)

Power Spectrum

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}[m]\, e^{-j\omega m} \qquad (1)$$

$$R_{xx}[m] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega m}\, d\omega \qquad (2)$$

(1) & (2) are called the Wiener-Khinchin Relations

Frequency Domain Analysis of LTI Systems

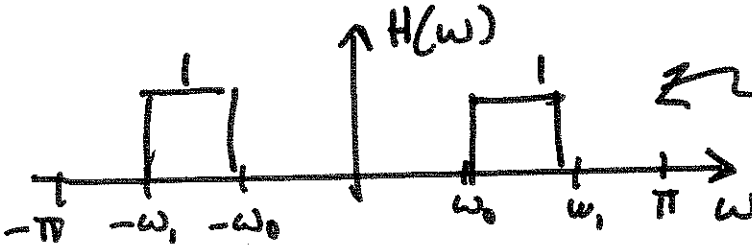
$$R_{yy}[m] = \sum_{\ell=-\infty}^{\infty} R_{xx}[m-\ell]\, C_{hh}[\ell]$$

$$\Longrightarrow \quad S_{yy}(\omega) = |H(\omega)|^2 \cdot S_{xx}(\omega)$$

Power spectrum of $X[n]$ is "shaped" by $|H(\omega)|^2$

Ex.

$$H(\omega) = \begin{cases} 1, & \omega_0 \le |\omega| \le \omega_1 \\ 0, & \text{otherwise} \end{cases}$$



ideal bandpass filter

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

In particular, the average output power is

$$E[\ {}^{2}[n]] = R_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_1}^{-\omega_0} S_{xx}(\omega)\, d\omega + \frac{1}{2\pi}\int_{\omega_0}^{\omega_1} S_{xx}(\omega)\, d\omega$$

This shows that we can interpret the area under $S_{xx}(\omega)$ for $\omega_0 \le |\omega| \le \omega_1$ as the average input power in that frequency band. Thus, $S_{xx}(\omega)$ can be viewed as a density function for power in the spectral (freq) domain.

Applications

Ex. $\boxed{\text{Noise Removal}}$

Suppose that we make noisy measurements of a random signal $s[n]$ in GWN $w[n]$:

$$x[n] = s[n] + w[n]$$

signal:

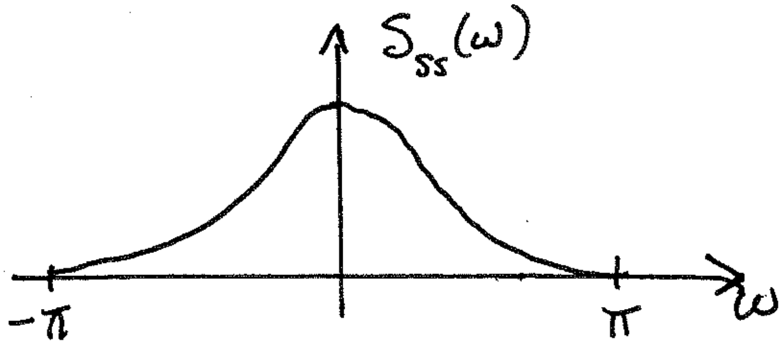
$m_s = 0$ $\quad$ $R_{ss}[m]$, $S_{ss}(\omega)$ arbitrary

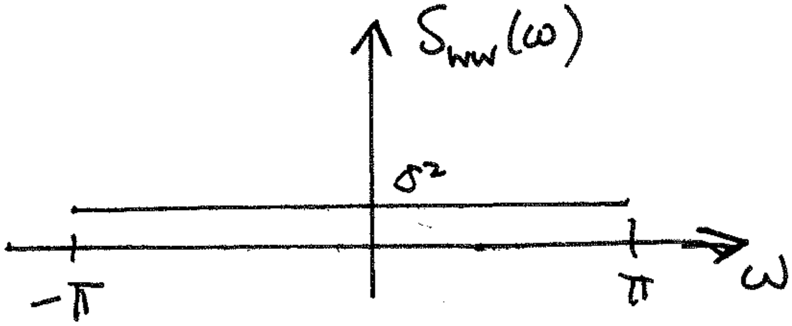
noise:

$m_w = 0$ $\quad$ $R_{ww}[m] = \sigma^2 \delta[m]$

$S_{ww}(\omega) = \sigma^2$ const.

Consider the case





Define the noise removal filter

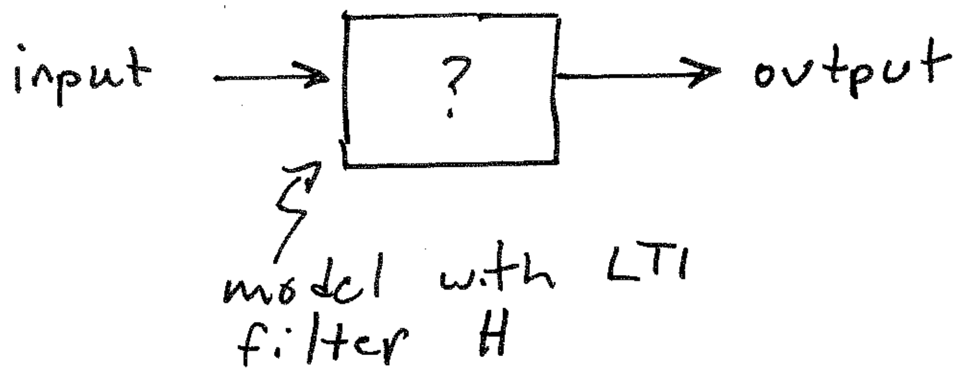
$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

How to choose ω_c?

Ex. System Identification

Suppose that we wish to model an <u>unknown</u> physical system with an LTI filter.



How can we "identify" the filter H?

White Noise Probing:

$$\text{input} \equiv x[n] = \text{GWN}$$

$$\Rightarrow \quad x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$m_y = 0$$

$$R_{yy}[m] = \sum_{k=-\infty}^{\infty} \sigma^2 h[k]\, h[m+k]$$

$$S_{yy}(\omega) = \sigma^2 \left| H(\omega) \right|^2$$

close, but only tells us magnitude

Stationary Processes

Properties of Correlation Function:

(1) $|R_{xx}[m]| \leq R_{xx}[0]$

(2) $R_{xx}[-m] = R_{xx}[m]$

(3) power of process $x[n]$ is given by $R_{xx}[0]$

proof of (1) : $(x[n] - x[n+m])^2 \geq 0$

$$\Rightarrow E\left[(x[n] - x[n+m])^2\right] \geq 0$$

similarly, $(x[n] + x[n+m])^2 \geq 0$

$$\Rightarrow E\left[(x[n] + x[n+m])^2\right] \geq 0$$

proof of (2) :

$$R_{xx}[m] = E\left[x[n]\,x[n+m]\right]$$

$$R_{xx}[-m] = E\left[x[n]\,x[n-m]\right]$$

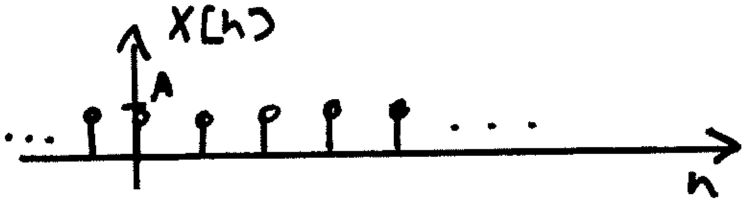
for any n!

take $n = n' - m$

$$R_{xx}[m] = E\left[x[n'-m]\,x[n']\right] = R_{xx}[-m]$$

Correlation

Total Correlation:



$$x[n] = A\,, \qquad P_A(a) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-a^2/2\sigma^2}$$

mean:

correlation function:

Zero Correlation:

Strong Correlation:

$$w[n] \longrightarrow \boxed{\frac{1}{1-a e^{-j\omega}}} \longrightarrow x[n]$$

GWN

$$x[n] = a\,x[n-1] + w[n] \quad ; \quad 0 < a < 1$$

impulse response $h[n] =$

mean:

Correlation function:

Stationarity

Let $x[n]$ be a random process.

Then we can talk about

$$Pr\left(|x[n]| < a\right)$$

or $$Pr\left(a < x[n] < b\right).$$

We can also study the joint probability distributions for two or more samples:

$$Pr\left(a < x[n] < b,\ c < x[n+m] < d\right)$$

or $$Pr\left(|x[n]| \le a,\ |x[n+1]| \le b,\ |x[n+2]| \le c\right)$$

Strict-Sense Stationarity (SSS)

Strict-sense stationarity means that these joint probability distributions are time-invariant.

ex. SSS

$$\Pr\left(a < x[n] < b,\ c < x[n+m] < d\right)$$
$$= \Pr\left(a < x[n+k] < b,\ c < x[n+k+m] < d\right)$$

for every $k \in \mathbb{Z}$.

SSS & LTI Systems

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

SSS

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$\Pr\left(a \le y[n] \le b\right) = \Pr\left(a \le \sum_k h[k]\, x[n-k] \le b\right)$$
$$= \Pr\left(a \le \sum_k h[k]\, x[n+m-k] \le b\right) \quad \text{by SSS of } x[n]$$
$$= \Pr\left(a \le y[n+m] \le b\right)$$

Wide-Sense Stationary (WSS)

(1) $E\left[x[n] \right] = m_x$ const.

(2) $E\left[x[n]\, x[n+m] \right] = R_{xx}[m]$

(1) $\Rightarrow$ average value of process is time-invariant

(2) $\Rightarrow$ average power of process and correlation between samples is time-invariant

WSS is <u>weaker</u> than SSS:

$$\text{SSS} \underset{\nLeftarrow}{\Rightarrow} \text{WSS}$$

WSS tells us that <u>average</u> behavior of single samples and products of samples is time-invariant.

SSS tells us that the joint probability distributions for arbitrary collections of samples are time-invariant.

WSS & LTI Systems

$$\underbrace{x[n]}_{\text{WSS}} \longrightarrow \underbrace{\boxed{H}}_{\text{LTI}} \longrightarrow y[n]$$

WSS, LTI $\Rightarrow$ WSS output

In most cases and applications WSS sufficient for analysis purposes.

(1) $E\left[x[n]\right] = m_x \Rightarrow E\left[y[n]\right] = m_y$ const.

(2) $\underbrace{E\left[x[n]\,x[n+m]\right] = R_{xx}[m]}$

correlation structure is sufficient to describe frequency content of input

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}[m]\, e^{-j\omega m}$$

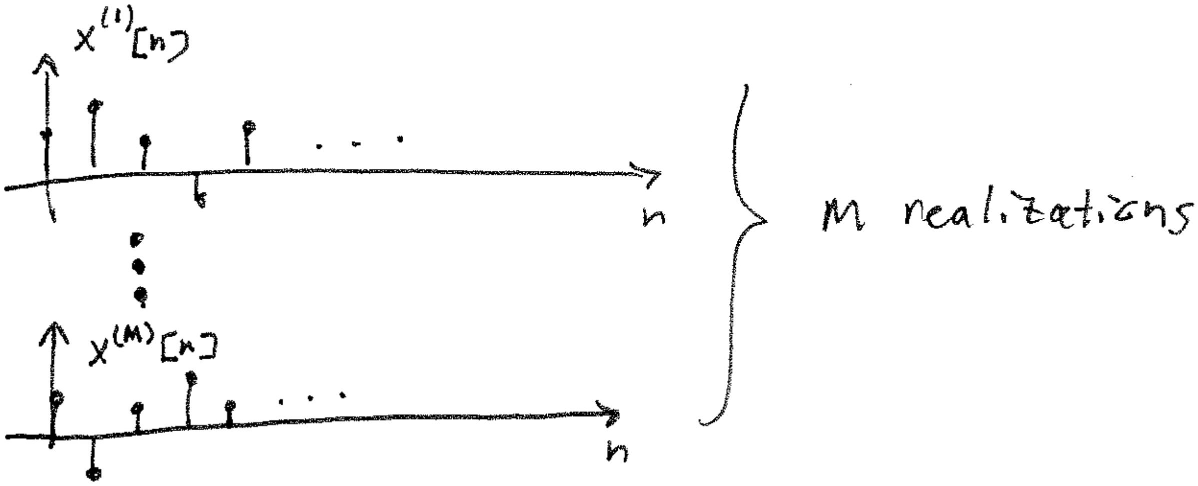
Thus, WSS is sufficient for frequency-domain analysis of random processes through LTI systems.

Estimating Means and Autocorrelations from Data

Suppose that we make several independent observations of a random process:

$$x^{(i)}[n], \qquad i = 1, \ldots, M$$

realizations



We can estimate the mean function using

$$\boxed{\hat{m}_x[n] = \frac{1}{M} \sum_{i=1}^{M} x^{(i)}[n]}$$

$$\hat{m}_x[n] \longrightarrow m_x[n] \quad \text{as } M \rightarrow \infty$$

We can estimate the autocorrelation function according to

$$\boxed{\hat{R}_{XX}[n, n+m] = \frac{1}{M} \sum_{i=1}^{M} x^{(i)}[n]\, x^{(i)}[n+m]}$$

If the process is <u>stationary</u>, then we can estimate the mean and auto correlation from a <u>single</u> realization using <u>time-averaging</u>.

$$\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\hat{R}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n]\, x[n+m]$$

$$\hat{m}_x \longrightarrow m_x$$

$$\hat{R}_{xx}[m] \longrightarrow R_{xx}[m] \qquad \text{as } N \longrightarrow \infty$$