DSP
ECE 431
COURSE NOTES

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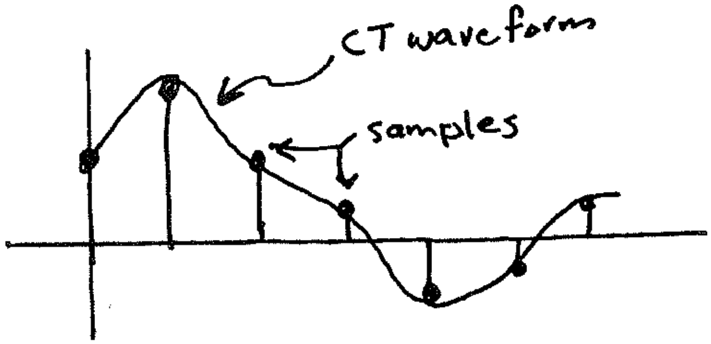
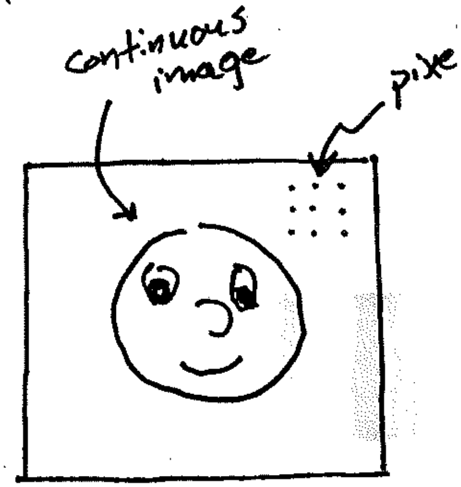
What is Digital Signal Processing (DSP)?

Basic idea:

Process signals and images
by performing operations
on <u>samples</u> of continuous-time
waveforms.



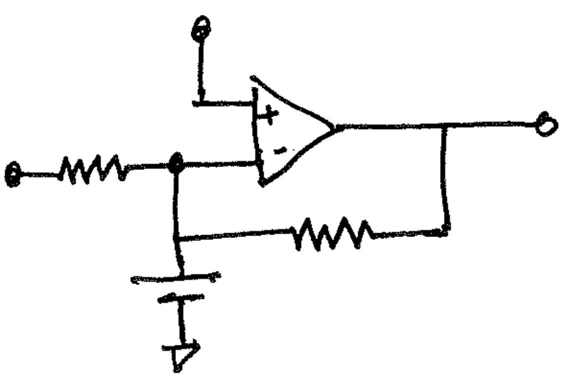


ex. compact disk,
cell phone

ex. ccd-based
digital camera,
medical imaging
systems

Motivation:

DSP takes signal processing out of
the realm of analog circuits...



RLC, op amps

<u>drawbacks</u>

noisy, sensitive to
environmental conditions,
fixed implementation
(non-programmable)

. . . and into the realm of the <u>Digital Computer</u> and microprocessors.

<u>advantages</u>
- fixed hardware (stable, acurate)
- adaptable processing (programmable)

Bits and programs replace electronic components and waveforms.

<u>Applications</u>

1. Communications
   - wireless phones, networks
   - modulation, equalization

2. Speech Processing
   - Analysis/synthesis vocoders for low bit-rate communication
   - speech recognition and synthesis

3. Image Processing

- coding and compression for images, streaming video, archival systems
- image restoration, pattern recognition, computer vision
- image reconstruction from projections (tomography)

4. Remote Sensing

- weather mapping / prediction
- satellite imaging, NASA
- hyperspectral imaging
- geophysical applications

5. Consumer Electronics

- cell phones
- CD players
- video games

6. Networks

- network traffic modeling, analysis and synthesis
- high-speed traffic shaping
- network monitoring and "bottleneck" prediction

A Brief History of DSP

Genesis: Numerical techniques for solving complicated equations

Newton, Euler, Bernoulli, Lagrange, Gauss, Fourier

1930's–40's WWII, radar, spread spectrum communications

1950's Analog signal processing, RLC's, tubes
First digital computers

1960's DSP driven by communications and oil industries, and space program

1965 Cooley-Tukey FFT (spark!)

1970's FFT algorithms
Digital filter design
Parks-McClellan filters (Rice, 1973)
First textbook (Oppenheim & Schafer, 1975)

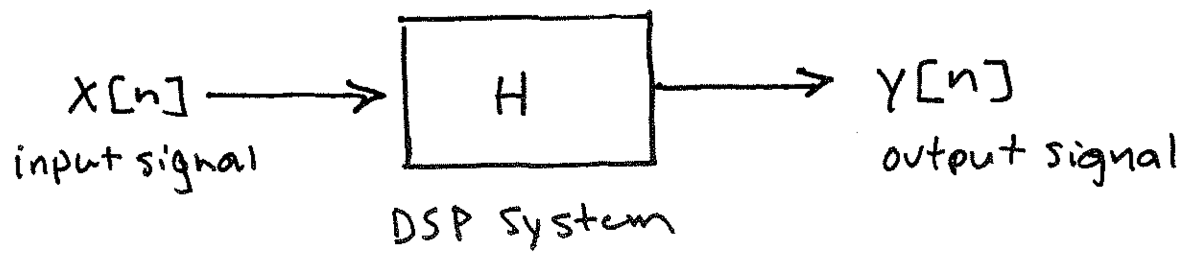
1980's Explosion of applications
CD's, remote sensing, medical imaging
communications...

1990's DSP is a standard technique
high-speed special purpose processors
for DSP (TI)
research and development driven
by communications and imaging

2000 and Beyond?

- Bioinformation Processing
- Network "Traffic Processing"
- Sensessors (sensor + processor)
- Nanoscale sensing and processing

Basic DSP Framework



ex. $x[n]$ ~ noisy speech signal

H ~ noise-reducing filter

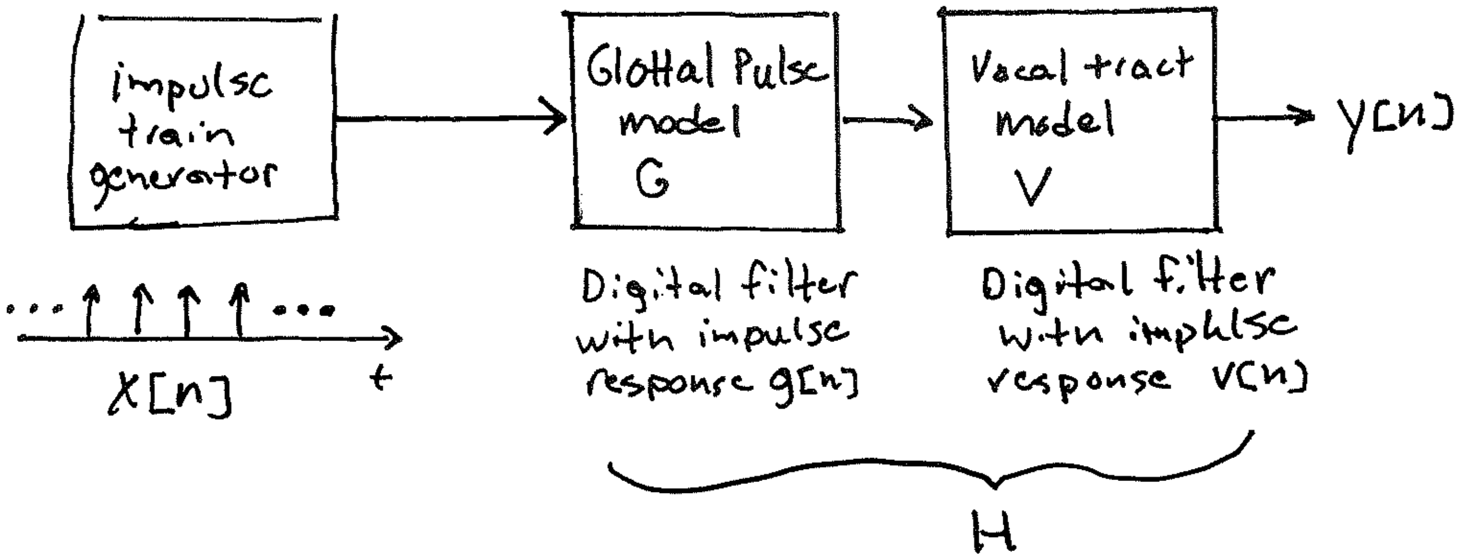
$y[n]$ ~ clean voice signal

DSP Problems

1. Given $x[n]$, $y[n]$, design or determine H
2. Given $x[n]$, H, compute $y[n]$
3. Given H, $y[n]$, find input $x[n]$

Speech Processing

Synthesis

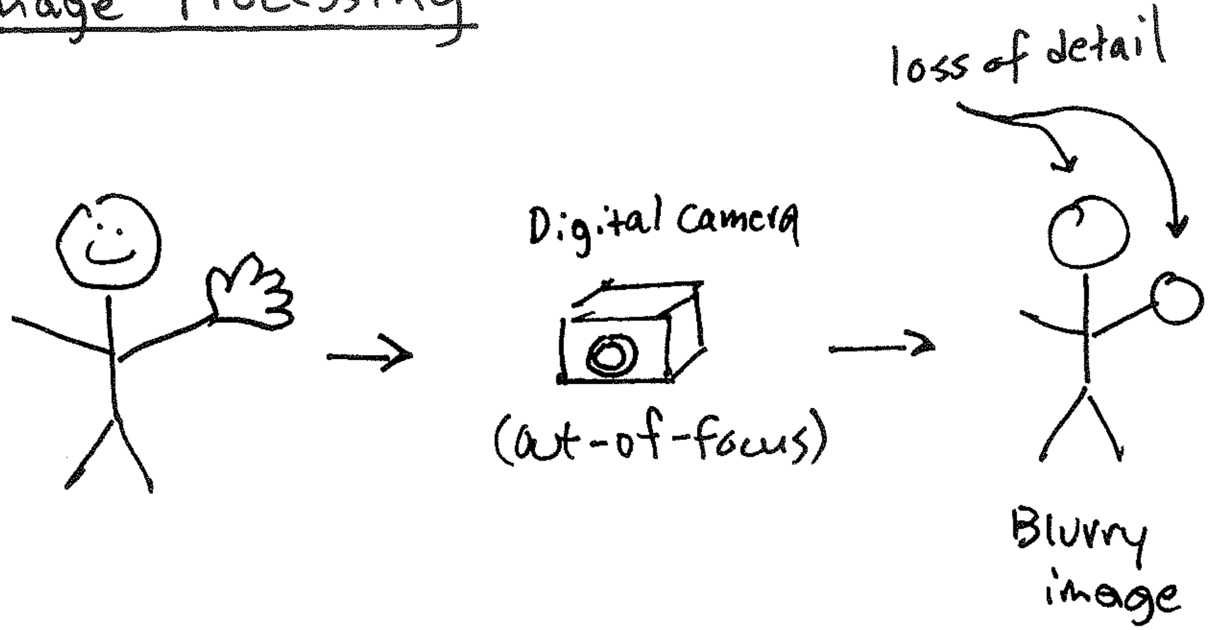


DSP Problem:

Design of digital filters $g[n]$ and $v[n]$.

Note: $\underbrace{h[n] = g[n] * v[n]}_{\text{overall filter}}$

Image Processing



DSP Problem:

- model blurring with a two-dimensional digital filter $b[n_1, n_2]$
- Compute inverse filter b^{-1} and apply it to blurry image

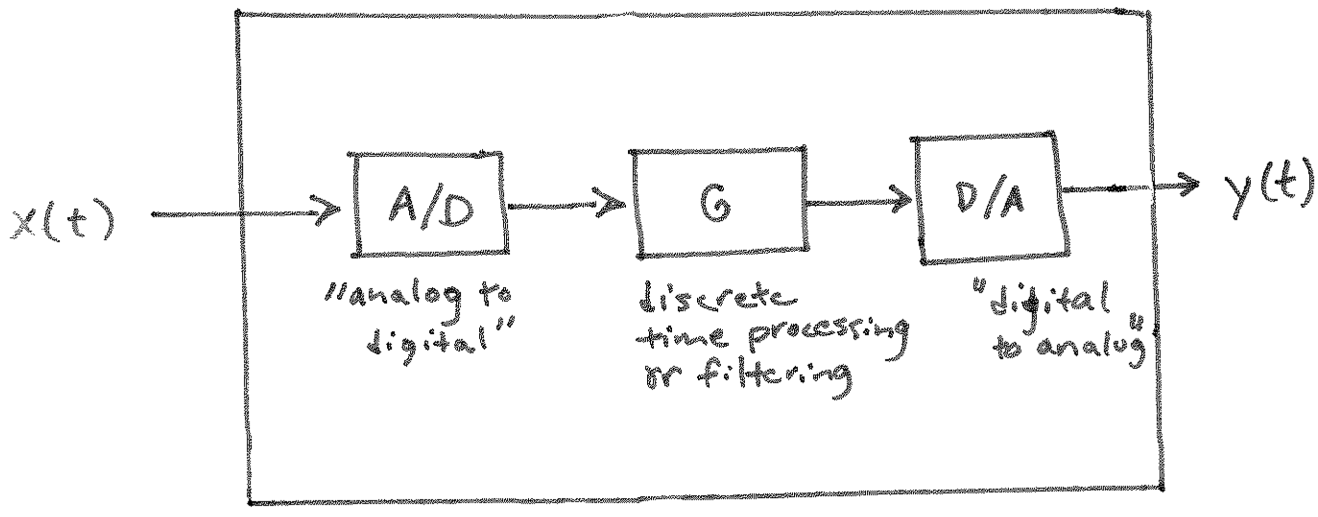
This problem is commonly called "image restoration"

Original Image

Blurry Image

Restored Image

The Basic DSP System



DSP System

Questions:

What is the overall effect of the DSP system?

How is the overall system (i.e., relationship between $x(t)$ and $y(t)$) related to G?

Can we design G to achieve a desired I/O characteristic $\frac{Y(\omega)}{X(\omega)}$?

Two-Dimensional Systems

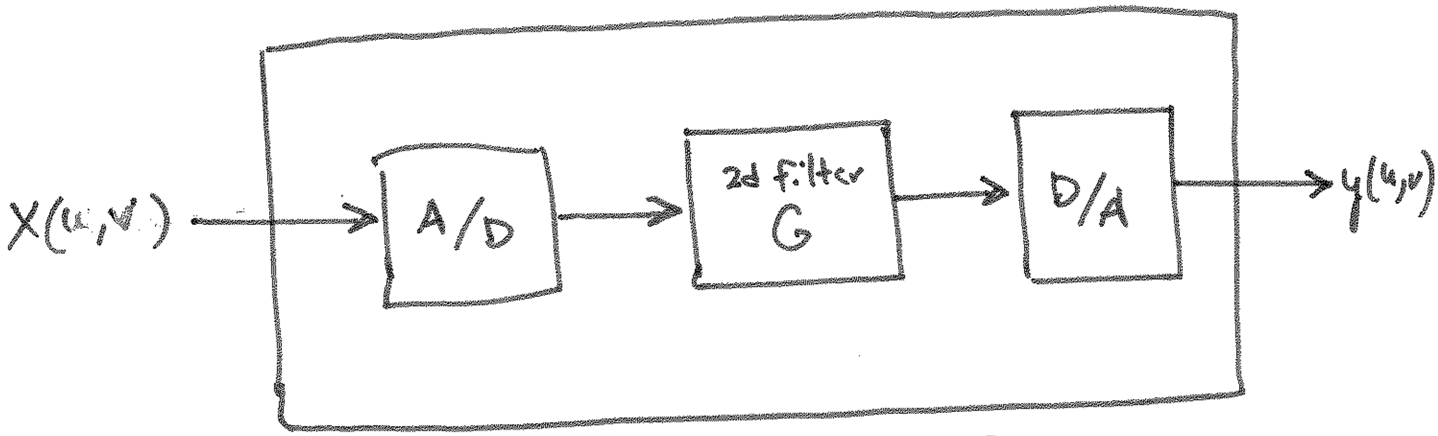


Image Processing System

① $x(u,v) \longrightarrow$ CCD camera $\longrightarrow x[m,n]$

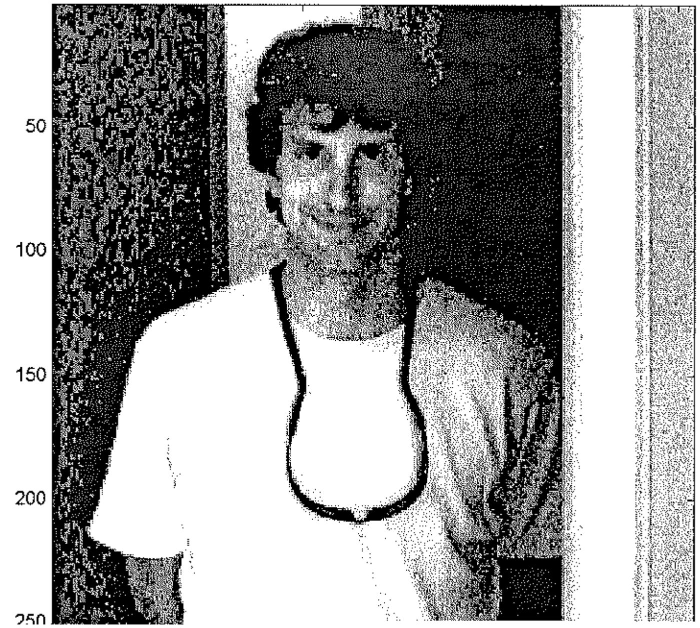
② linear filtering

$$y[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g[k,l]\, x[m-k,\, l-n]$$

2D convolution

③ Display on screen/computer

$$y[m,n] \longrightarrow y(u,v)$$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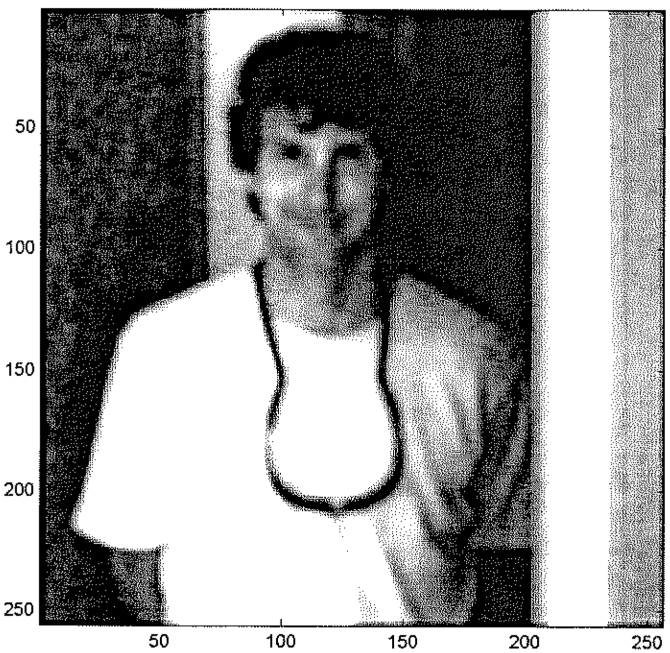
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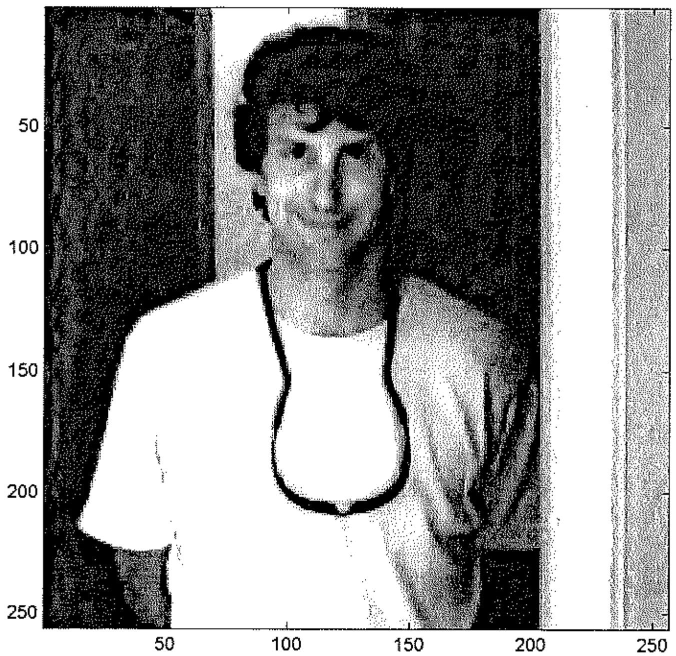
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SET A: Review of Continuous-Time Signals & Systems

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ECE 330

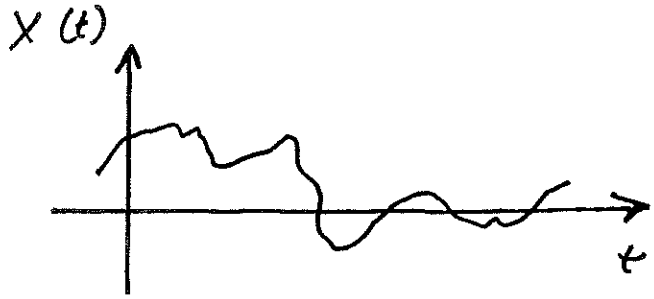
Outline

- systems
- signals
- LTI Systems
- convolution
- differential equations
- Fourier transforms

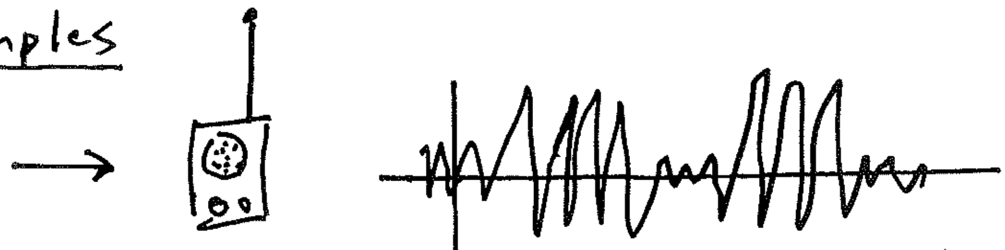
* We will see many parallels between continuous-time and discrete-time

Definition: A <u>continuous-time signal</u> is a function of a real variable t

$$x(t), \quad t \in \mathbb{R} \quad (-\infty < t < \infty)$$
$$\text{or} \quad t \in [0,T] \quad (0 \le t \le T)$$



<u>Examples</u>



AM radio signal

Others?

Definition: A continuous-time system is a transformation that maps an input signal $x(t)$ into an output $y(t)$:

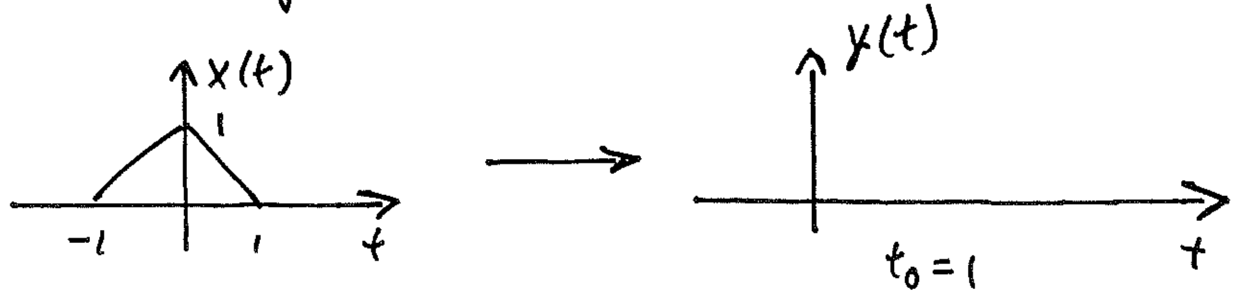
$$y(t) = T\{x(t)\}$$

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t)$$

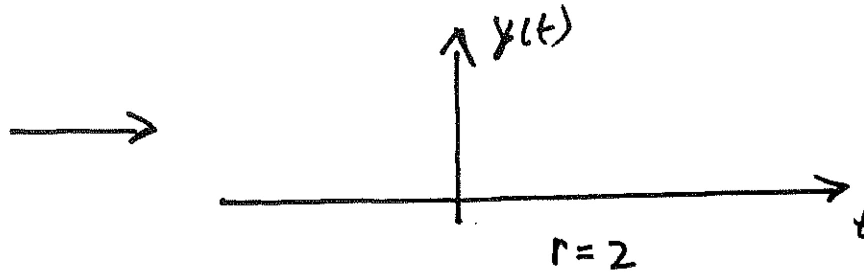
transformation $\equiv$ a rule or formula for computing $y(t)$ given $x(t)$

Examples: ☆ Reader ☆

Delay: $y(t) = x(t - t_0)$, $t_0 \in \mathbb{R}$



Scaling: $y(t) = x(t/r)$, $r \neq 0$



★ Reader ★

<u>Definition</u>: A system is <u>linear</u> if

1.

2.

<u>Definition</u>: A system is time-invariant/ shift-invariant if

★ Linear time-invariant systems are completely described by their <u>impulse</u> <u>responses</u> :

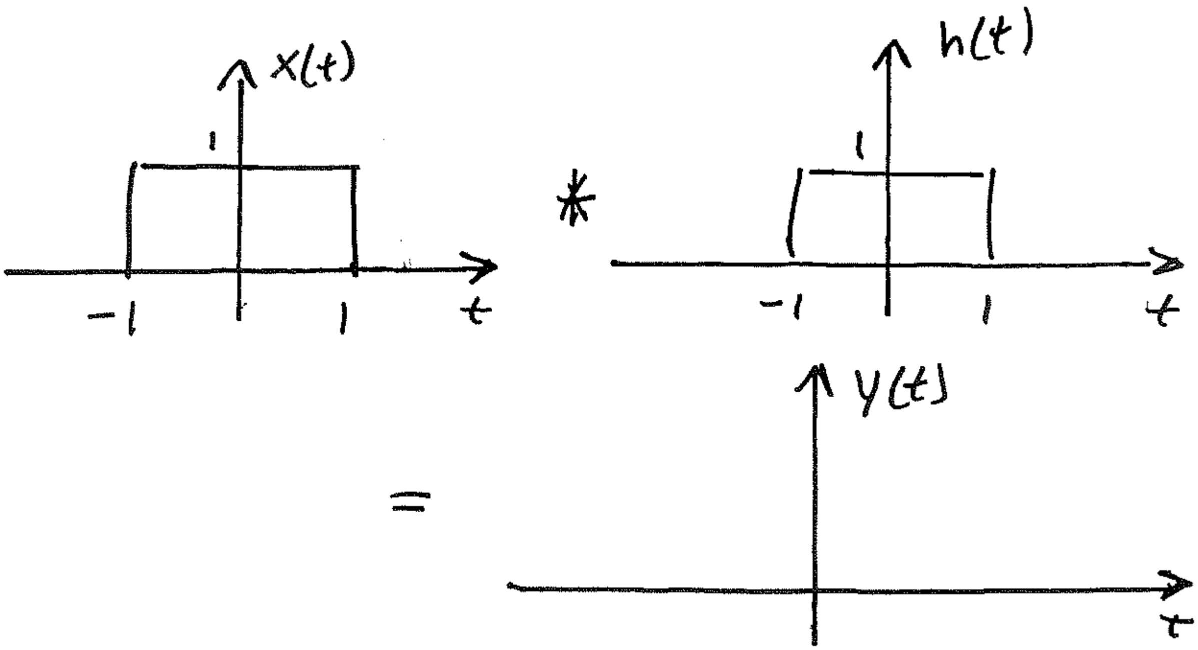
$$\delta(t) \longrightarrow \boxed{T} \longrightarrow h(t)$$

Dirac deltafunction

★ Given an input $x(t)$ to an LTI system with impulse response, $y(t)$, the output, can be computed using the <u>convolution</u> <u>integral</u>:

$$y(t) = x(t) * h(t)$$
$$= (x * h)(t)$$
$$\equiv \int_{-\infty}^{\infty} x(u)\, h(t-u)\, du$$
$$= \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$$

<u>Example:</u> ★ Reader ★



A5

★ A large class of linear systems consists of those whose input $x(t)$ and output $y(t)$ obey an N-th order <u>linear constant coefficient differential equation</u>:

$$\sum_{k=0}^{N} a_k \underbrace{y^{(k)}(t)}_{\substack{k\text{-th} \\ \text{derivative}}} = \sum_{\ell=0}^{M} b_\ell \, x^{(\ell)}(t)$$

<u>Example</u>:

$$2\frac{dy}{dt} + y = \frac{1}{2}\frac{dx}{dt} + 7x$$

Given $x(t)$, solve for $y(t)$.

⟶ analog computers

Mathematical Solution Methods:

1. Homogeneous equation
   particular solution
   combine

2. Laplace transforms

<u>Definition:</u> Continuous-Time Fourier Transform (CTFT)

$$(1) \quad X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$(2) \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)\, e^{+j\Omega t}\, d\Omega$$

<u>Remark:</u>

$$x(t) \xleftrightarrow{\text{CTFT}} X(\Omega)$$

$x(t)$: continuous $t \in \mathbb{R}$, aperiodic

$X(\Omega)$: Continuous $\Omega \in \mathbb{R}$, aperiodic

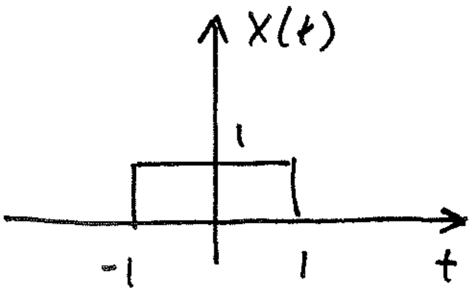
☆ <u>Interpretation:</u> The inverse CTFT in (2) can be viewed as a linear superposition of infinitesimally small complex sinusoids

$$\underbrace{\frac{1}{2\pi} X(\Omega)\, d\Omega}_{\text{amplitude}} \cdot \underbrace{e^{j\Omega t}}_{\text{complex sinusoid}}$$

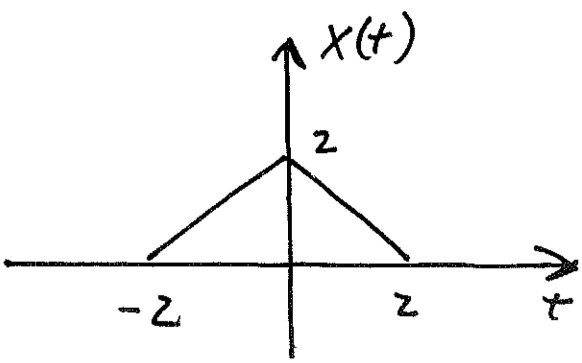
Example: ☆ Reader ☆

$$x(t) = e^{j\Omega_0 t} \xleftrightarrow{\text{CTFT}} X(\Omega) =$$

$$x(t) = \cos \Omega_0 t \xleftrightarrow{\text{CTFT}} X(\Omega) =$$



$$\xleftrightarrow{\text{CTFT}} X(\Omega) =$$



$$\xleftrightarrow{\text{CTFT}} X(\Omega) =$$

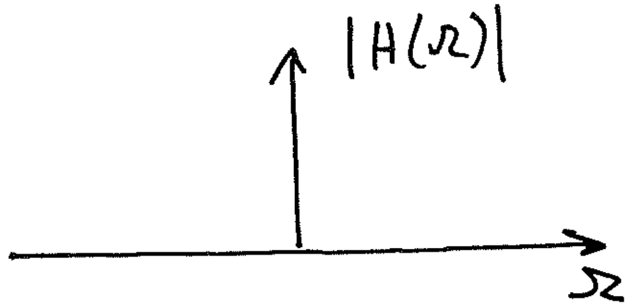
CTFT and Convolution

$$\underset{\text{convolution}}{x(t) * h(t)} \xleftrightarrow{\text{CTFT}} \underset{\text{product}}{X(\Omega) \cdot H(\Omega)}$$

Reader

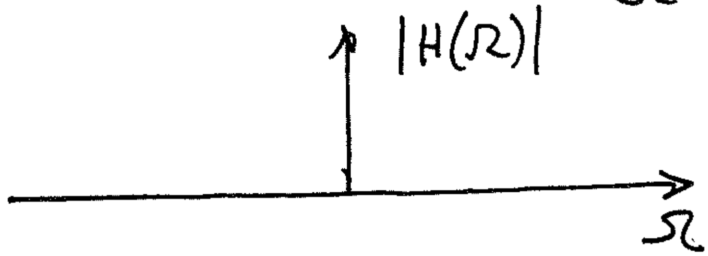
Filters

Lowpass

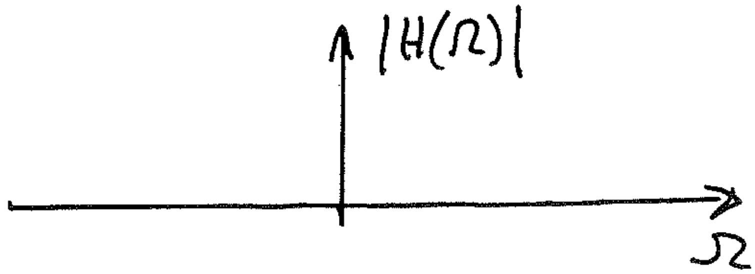


Highpass

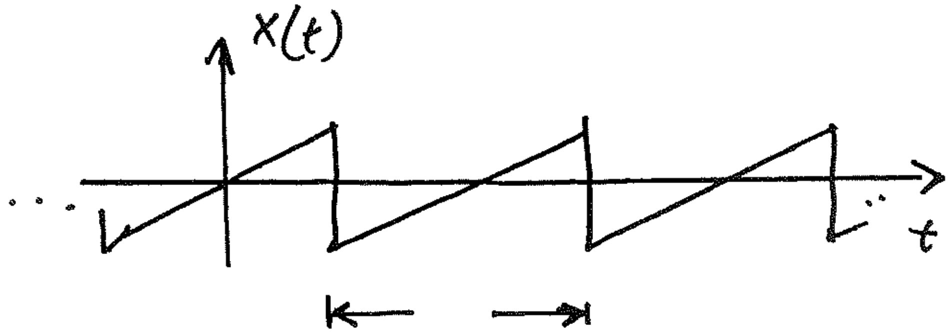
Bandpass



Bandstop



<u>Definition:</u> Continuous-time Fourier Series (CTFS)



$$x(t-T) = x(t)$$

$$x(t+kT) = x(t), \quad k \in \mathbb{Z}$$

(k integer-valued)

$$c_k = \int_0^T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

$$\Omega_0 \equiv \frac{2\pi}{T} = \text{fundamental frequency}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2\pi}{T}kt}$$

<u>Remark:</u>

$$x(t) \xleftrightarrow{\text{CTFS}} c_k$$

continuous $t \in \mathbb{R}$, periodic — discrete $k \in \mathbb{Z}$, aperiodic

Sampling and Reconstruction

Notation:

$x_c(t)$ = CT signal

CTFT:

$$X_c(\Omega) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j\Omega t}\, dt$$

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega t}\, d\Omega$$

$x[n]$ = DT signal

DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

In order to process CT signals effectively with a computer, we need to know how to <u>sample</u>.

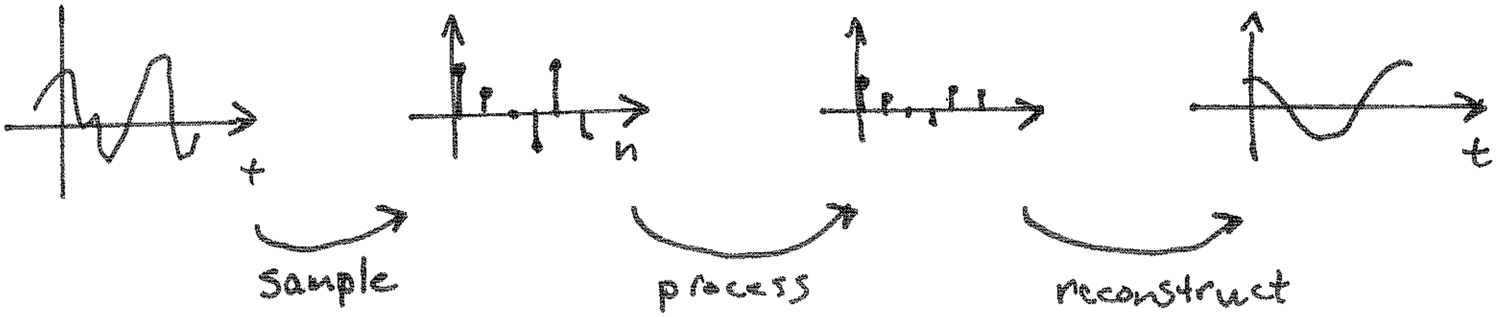
$$x_c(t) \longrightarrow \boxed{A/D} \longrightarrow x[n]$$

and <u>reconstruct</u>

$$x[n] \longrightarrow \boxed{D/A} \longrightarrow x_c(t)$$

Then we can process CT signals by:

$$x_c(t) \longrightarrow \boxed{A/D} \xrightarrow{x[n]} \underset{\text{LTI}}{\overset{\text{DT}}{\boxed{H}}} \xrightarrow{y[n]} \boxed{D/A} \longrightarrow y_c(t)$$

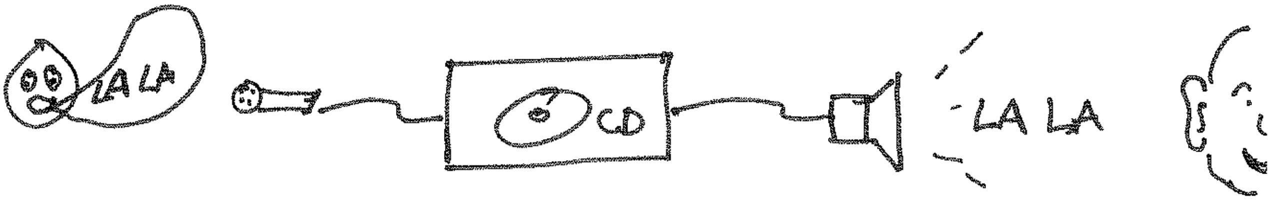


We'll talk about sampling and reconstruction first, then look at processing.

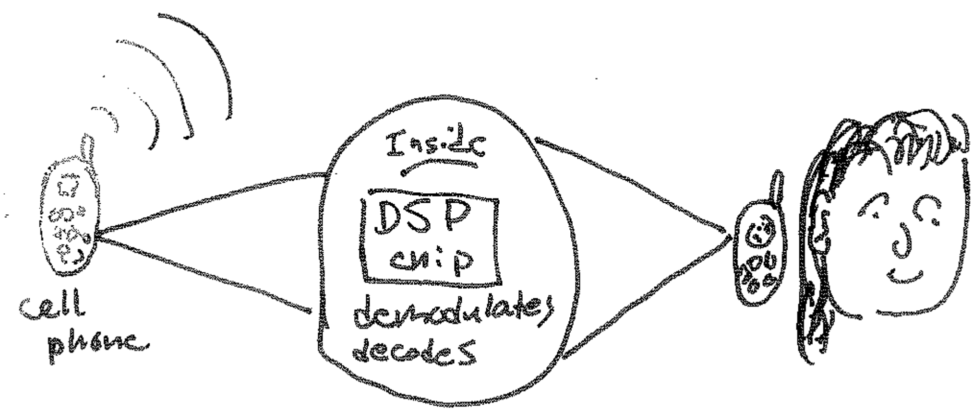
Sampling – The BIG Picture

Processing analog (CT) signals on digital computers is a huge application area.

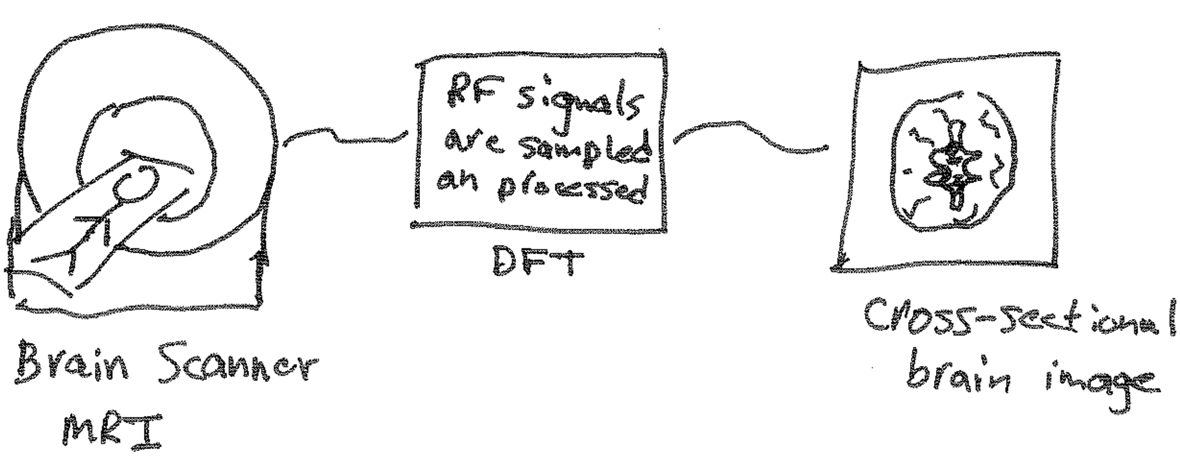
Ex.



Ex.

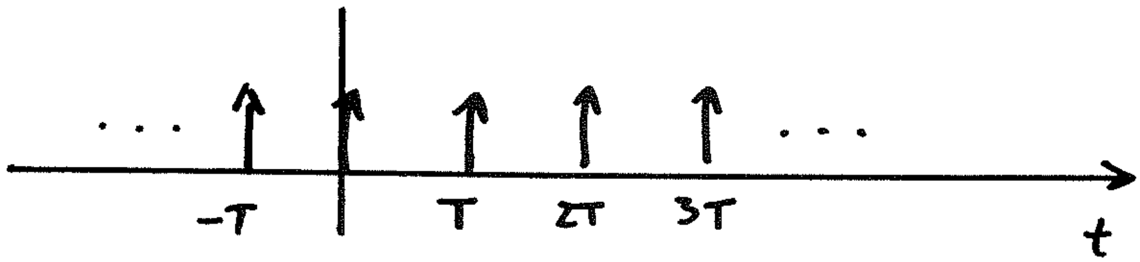


Ex.



Impulse Train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Define

$$x_s(t) = x_c(t) \cdot s(t)$$

$$= x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT).$$

★ KEY POINT ★

$$x_s(t) = \sum_n x[n]\,\delta(t-nT)$$

We have a one-to-one correspondence between $x_s(t)$ and $x[n]$.

Given $x_s(t)$, we know $x[n]$

Given $x[n]$ (and T), we know $x_s(t)$

Impulse sampled $x_s(t)$ is the CT representation of the DT signal $x[n]$.

FREQUENCY DOMAIN INTERPRETATION OF IDEAL SAMPLING

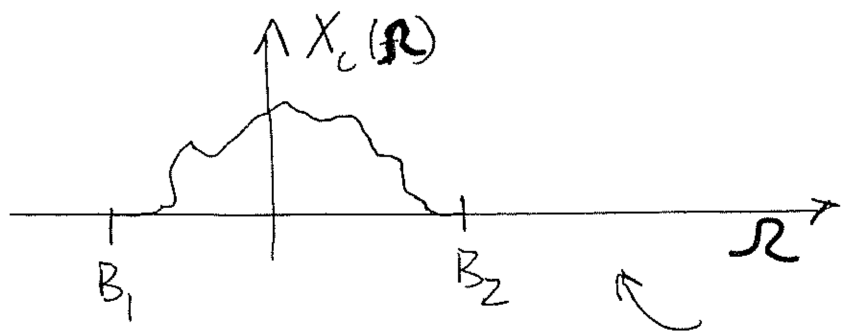
Preliminaries

(1) Bandlimited CT Signals

Defn: A CT signal $x_c(t)$ is bandlimited to the band $[B_1, B_2]$ if

$$X_c(\Omega) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j\Omega t}\, dt$$

is exactly zero for all $\Omega \notin [B_1, B_2]$



Zero for frequencies outside the band

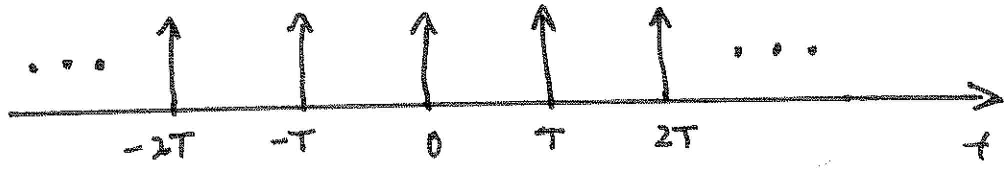
(2) Recall the FT pair

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \longleftrightarrow S(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

periodic impulse train in time

impulse train in frequency

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



periodic!

Periodic CT signal $\Rightarrow$ <u>Fourier Series</u>

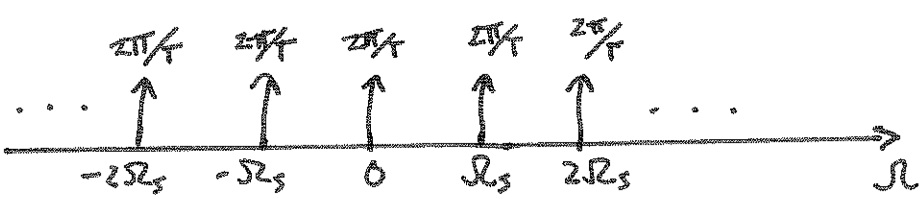
Fourier Series coefficients:

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} \delta(t)\, e^{-jk\Omega_s t}\, dt$$

$$= \frac{1}{T}$$

$$s(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{-jk\Omega_s t}$$

$$\Rightarrow \quad S(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi \underbrace{\delta(\Omega - k\Omega_s)}_{\leftarrow \text{FT of } e^{-jk\Omega_s t}}$$



Recall (301)

$$x(t)\cdot y(t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi} X(\Omega) * Y(\omega)$$

proof:

$$\int \left(x(t)\cdot y(t)\right)\cdot e^{-j\Omega t}\,dt$$

$$= \int \left(\frac{1}{2\pi}\int X(\Omega')e^{j\Omega' t}\,d\Omega'\right)\cdot y(t)\cdot e^{-j\Omega t}\,dt$$

$$= \frac{1}{2\pi}\int X(\Omega')\left(\int y(t)e^{-j(\Omega-\Omega')t}\,dt\right)d\Omega'$$

$$= \frac{1}{2\pi}\int X(\Omega')\,Y(\Omega-\Omega')\,d\Omega'$$

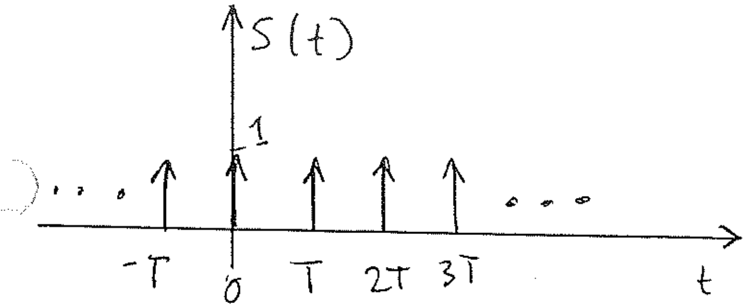
Implication:

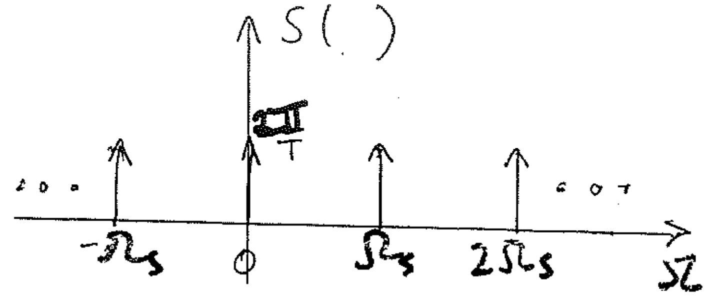
$$x_s(t) = x_c(t)\cdot s(t)$$

$$\Big\updownarrow \text{CTFT}$$

$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)$$





$f_s = \frac{1}{T} \qquad \Omega_s = \frac{2\pi}{T}$

Ideal Sampling

Recall

$$x_s(t) = x_c(t) \cdot s(t)$$

remember
1-1 correspondence
with DT $x[n]$

Consider the CT FT of $x_s(t)$

$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)\, S(\Omega - \alpha)\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\alpha) \frac{2\pi}{T} \sum_k \delta(\Omega - \alpha - k\Omega_s)\, d\alpha$$

$$= \frac{1}{T} \sum_k \int_{-\infty}^{\infty} X_c(\alpha)\, \delta(\Omega - \alpha - k\Omega_s)\, d\alpha$$

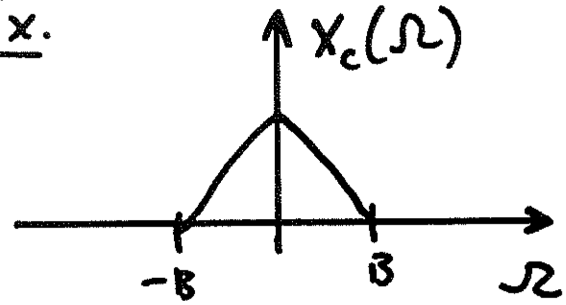
$$= \frac{1}{T} \sum_k X_c(\Omega - k\Omega_s)$$

FUNDAMENTAL RESULT

$$\boxed{X_s(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)}$$

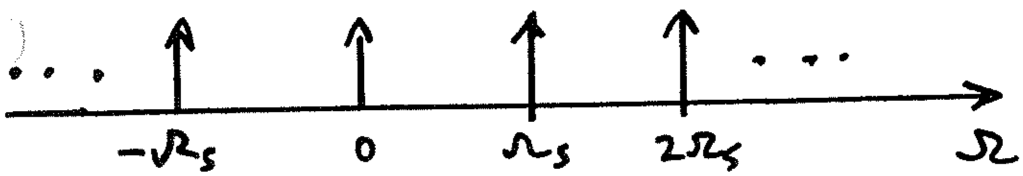
$\Rightarrow$ $X_c(\Omega)$ consists of periodically repeated copies of $X_c(\Omega)$.

Ex.



Fourier transform of $x_c(t)$.
Bandlimited

$*$



Fourier Transform of $s(t)$ with period T.
$\Omega_s = \frac{2\pi}{T}$

$\Omega_s - B > B$

$\Rightarrow \boxed{\Omega_s > 2B}$

$\Omega_s - B < B$

$\Rightarrow \boxed{\Omega_s < 2B}$

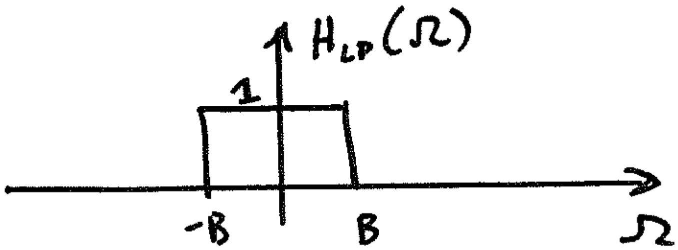
Summary:

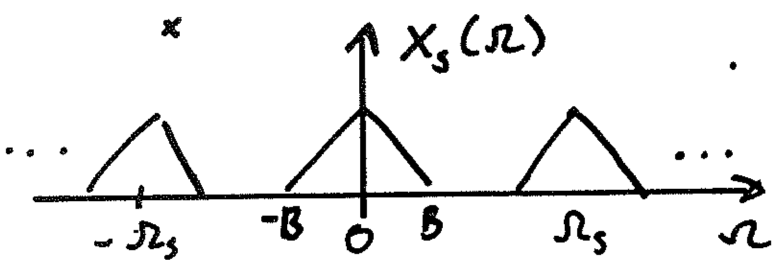
If $\Omega_s \geq 2B$, then we simply <u>copy</u> the bandlimited spectrum $X_c(\Omega)$ every $\Omega_s = \frac{2\pi}{T}$ in frequency.

<u>Furthermore</u>, we can reconstruct $x_c(t)$ by lowpass filtering $x_s(t)$; passing $x_s(t)$ through an ideal lowpass filter.

$$x_c(t) \longrightarrow \otimes \xrightarrow{x_s(t)} \boxed{H_{LP}} \longrightarrow x_c(t)$$

$$s(t) = \sum_n \delta(t - nT)$$





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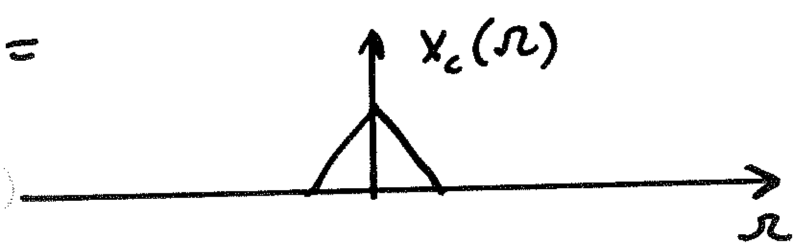
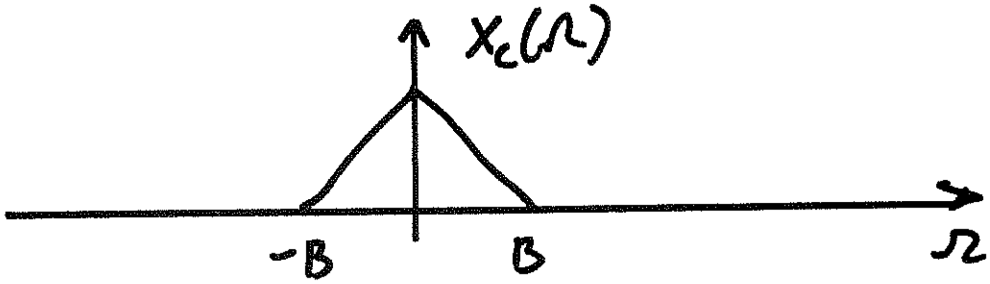
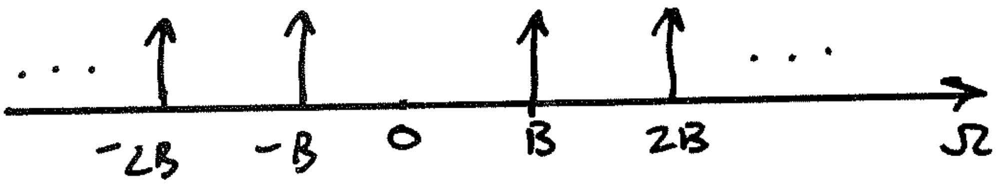


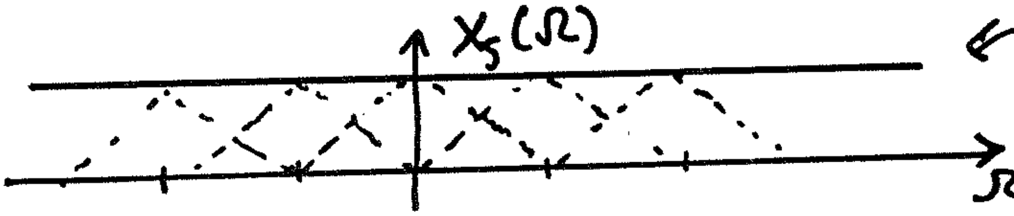
We can perfectly reconstruct a bandlimited signal from its samples using an ideal lowpass filter!

If $\Omega_s < 2B$, then the copies of $X_c(\Omega)$ <u>overlap</u> and their sum doesn't resemble copies of $X_c(\Omega)$.



extreme case - $\boxed{\Omega_s = B}$



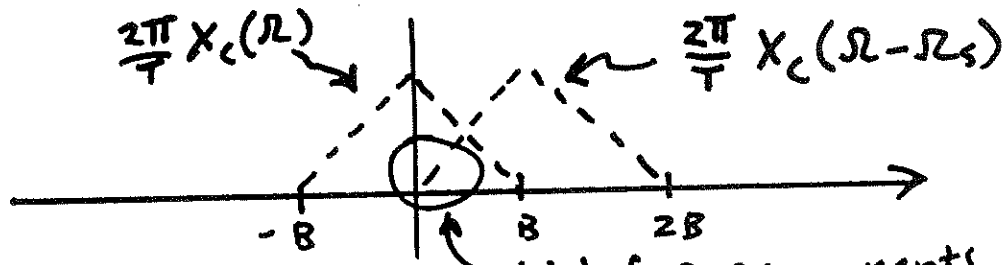


constant!

No amount of filtering can recover $X_c(\Omega)$!

$$\Omega_s < 2B \Rightarrow \underline{\text{Aliasing}}$$

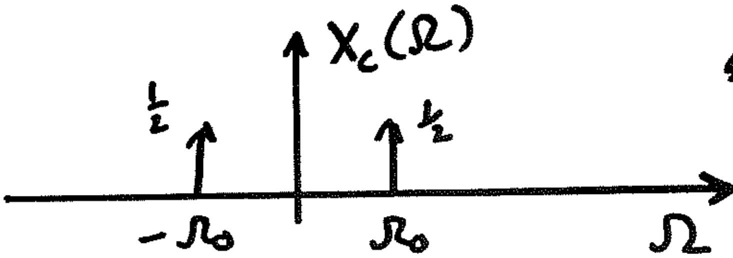
That is, high frequencies in $X_c(\Omega - \Omega_s)$ <u>acting</u> like low frequencies in $X_c(\Omega)$ cop



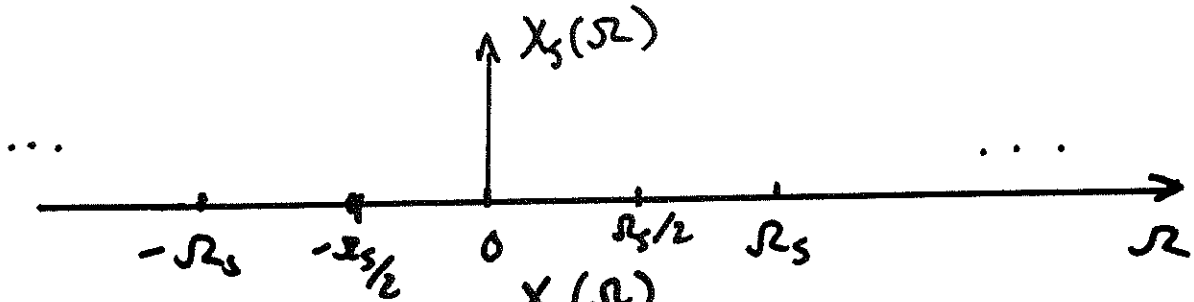
high freq. components acting or "aliasing" as low freq. components

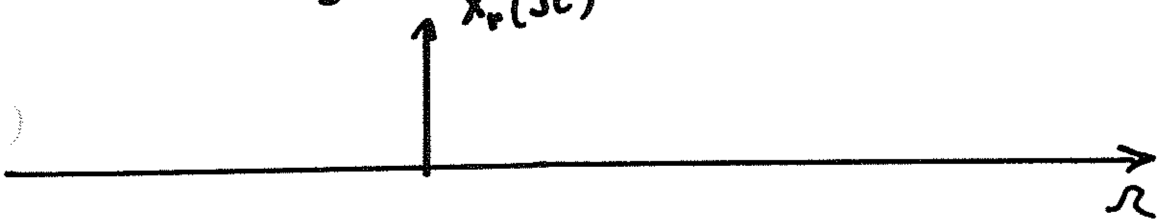
Example

$$x_c(t) = \cos(\Omega_0 t)$$

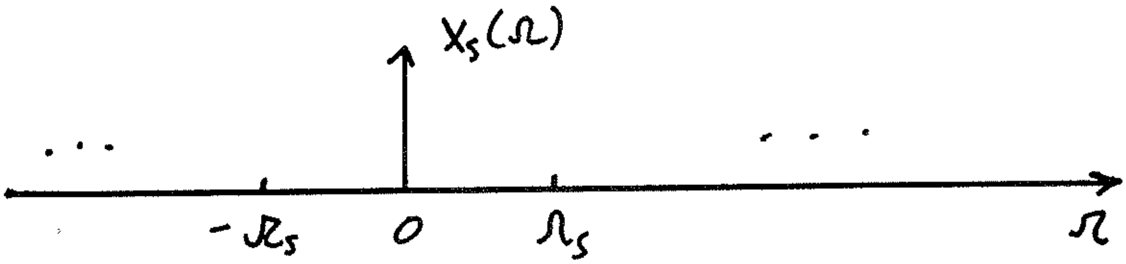


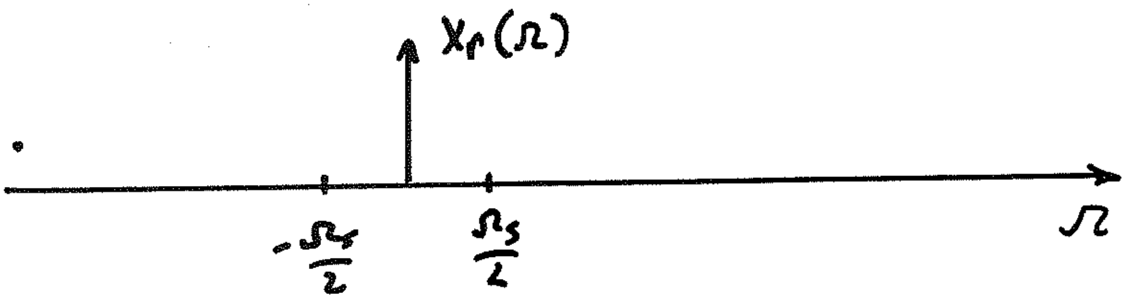
bandlimited to $[-\Omega_0, \Omega_0]$





$\Omega_s > 2\Omega_0$



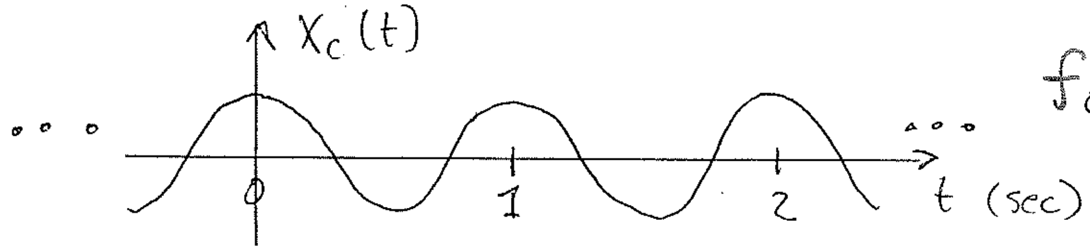


$\Omega_s < 2\Omega_0$

$$x_r(t) = \cos(\qquad\qquad)$$

Physical Interpretation:

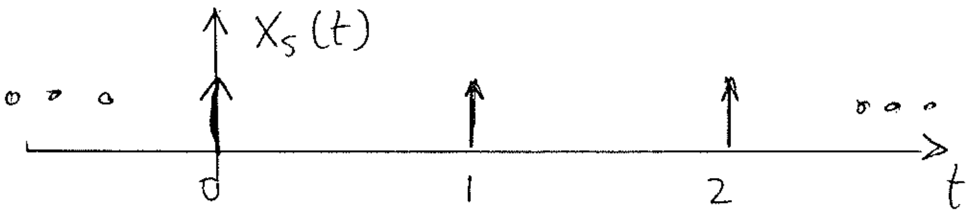
If $\Omega_s < 2\Omega_0$, then $x_c(t)$ "wiggles" several times between samples



$f_0 = 1$ Hz

$\Omega_0 = 2\pi$ rad/s

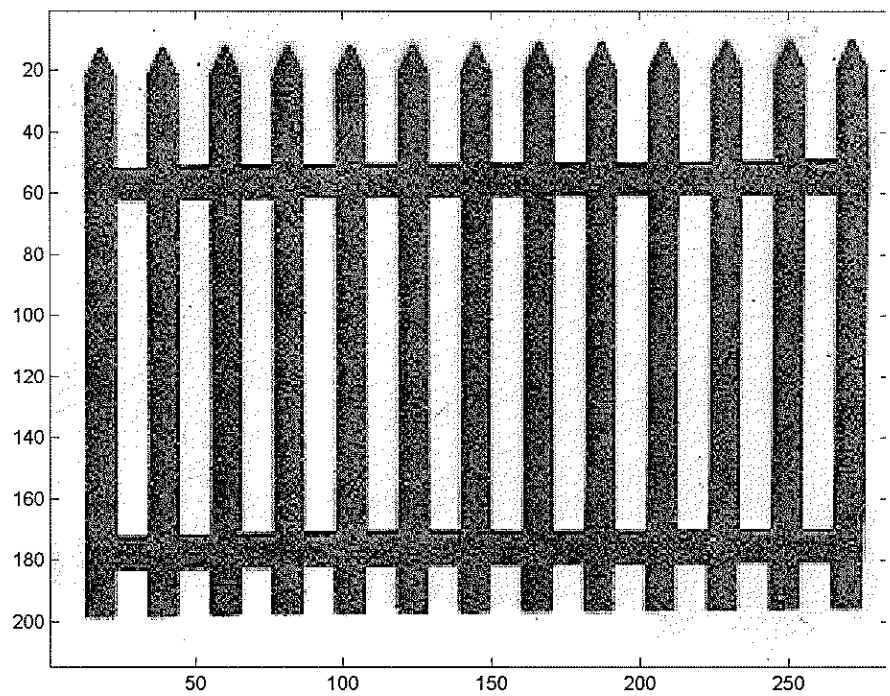
Set $T = 1$



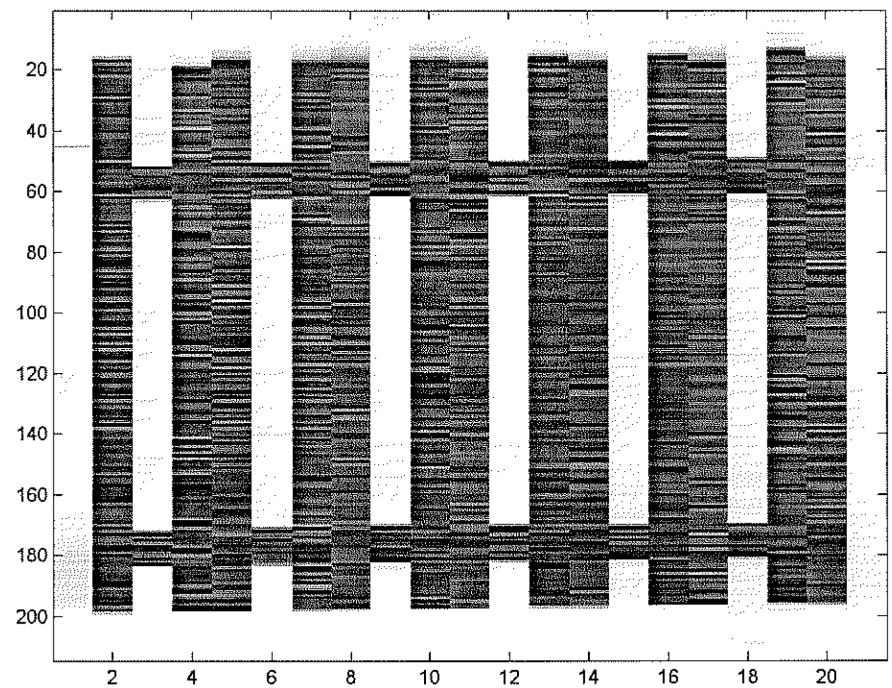
How can you reconstruct a sine wave from that?

Samples spaced too far apart to see wiggliness of true signal!

Aliasing in 2D



20 samples/cycle



1.4 samples/cycle

Nyquist Theorem

Let $x_c(t)$ be a bandlimited CT signal with

$$X_c(\Omega) = 0 \quad \text{for } |\Omega| > B.$$

Then $x_c(t)$ is completely determined by its samples $x[n] = x_c(nT)$, $n \in \mathbb{Z}$, if

$$\Omega_s = \frac{2\pi}{T} > 2B$$

B = "Nyquist Frequency"

$2B$ = "Nyquist Rate"

twice the highest frequency component in $x_c(t)$

This is why CDs are sampled at $f_s = 44.1$ kHz — because people can only hear up to ≈ 20 kHz.

Sampling & Reconstruction (IDEAL)

TIME | FREQUENCY