Laplace Transform and Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\Big\updownarrow \mathcal{L}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

roots give location of zero

roots give locations of poles

Z-transform and Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Big\updownarrow Z$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum^{N} a_k z^{-k}}$$

Laplace Transform

$x(t)$ = CT signal

$s = \sigma + j\omega$ (complex number) (cartesian coordinates)

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

CTFT

$$X(\Omega) = X(s)\Big|_{s=j\Omega}$$

Inv Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

hold σ fixed & integrate ω from $+\infty$ to $-\infty$ (with $\sigma = 0$ this is inv CTFT)

Differentiation

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$$

proof: $\frac{d}{dt}\frac{1}{2\pi j}\int X(s) e^{st} ds = \frac{1}{2\pi j}\int sX(s) e^{st} ds$

more generally, k-th derivative

$$x^{(k)}(t) \overset{\mathcal{L}}{\longleftrightarrow} s^k X(s)$$

Z-transform

$x[n]$ = DT signal

$z = re^{j\omega}$ (complex number polar coord)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

DTFT

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

Inv Z-transform

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$$

contour integration

Time Delay

$$x[n-1] \overset{Z}{\longleftrightarrow} z^{-1} X(z)$$

k unit delay

$$x[n-k] \overset{Z}{\longleftrightarrow} z^{-k} X(z)$$