

Laplace Transform

$X(t)$ = CT signal

$$S = \sigma + j\omega \quad \left(\begin{array}{l} \text{Complex} \\ \text{number} \\ \text{(Cartesian} \\ \text{coordinates)} \end{array} \right)$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

CTFT

$$X(\omega) = X(s) \Big|_{s=j\omega}$$

Inv Laplace Transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

← hold σ fixed & integrate ω from $+\infty$ to $-\infty$
(with $\sigma > 0$ this is inv CTFT)

Differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} s X(s)$$

proof: $\frac{d}{dt} \frac{1}{2\pi j} \int X(s) e^{st} ds = \frac{1}{2\pi j} \int s X(s) e^{st} ds$

more generally, k -th derivative

$$x^{(k)}(t) \xleftrightarrow{\mathcal{L}} s^k X(s)$$

Z-transform

$X[n]$ = DT signal

$$z = r e^{j\omega} \quad \left(\begin{array}{l} \text{Complex} \\ \text{number} \\ \text{(polar coord)} \end{array} \right)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

DTFT

$$X(\omega) = X(z) \Big|_{z=e^{j\omega}}$$

Inv Z-transform

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

← contour integration

Time Delay

$$x[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1} X(z)$$

k unit delay

$$x[n-k] \xleftrightarrow{\mathcal{Z}} z^{-k} X(z)$$

Laplace Transform and Differential Equations

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

$\updownarrow \mathcal{L}$

$$\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^M b_k s^k X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

roots give location of zero

roots give location of poles

Z-transform and Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$\updownarrow \mathcal{Z}$

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$