ECE 901
Lecture 16: Wavelet Approximation Theory

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1 Introduction

In Lecture 4 and 15, we investigated the problem of denoising a smooth signal in additive white noise. In Lecture 4, we considered Lipschitz functions and showed that by filling constants on a uniform partition of width $n^{-1/3}$ we can achieve an $n^{-2/3}$ rate of MSE convergence.

In Lecture 15, we considered Holder-α smooth functions, and we demonstrated that by automatically selecting partition width and using polynomial fits we can obtain a MSE convergence rate of $n^{-2\alpha/2\alpha+1}$, substantially better when $\alpha > 1$. Perhaps more important is the fact that we don't need to know the value of α a priori. The estimator $\widehat{f}_n$ is fundamentally different than its counterpart in Lecture 4.

In both cases $\widehat{f}_n(t)$ is a linear function (polynomial on constant fit) of the data in each interval of the underlying partition. In Lecture 4, the partition was independent of the data, and so the overall estimator is a **linear function of the data**.

However, in Lecture 15 the partition itself was selected based on the data. Consequently, $\widehat{f}_n(t)$ is a **non-linear function of the data**. Linear estimators (linear functions of the data) cannot adapt to unknown degrees of smoothness. In this lecture, we lay the groundwork for one more important extension in the denoising application - spatial adaptivity. That is, we would like to construct estimators that not only adapt to unknown degrees of global smoothness, but that also adapt to **spatially varying** degrees of smoothness.

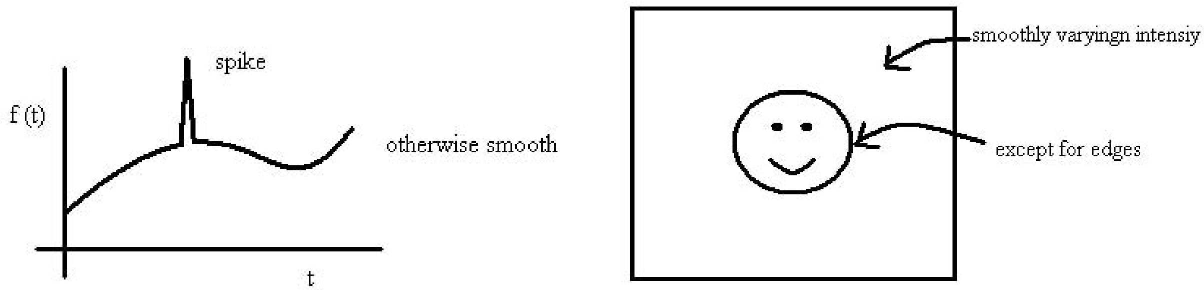
As discussed before the key for good estimation performance is the use of models capable of represented the target concept of interested (be it a regression function or a classification rule) using a small number of "numbers" (parameters). In this lecture we will focus solely on function approximation procedures, and will use these results in the next lecture to derive simple versions of the current state-of-the-art in denoising and non-parametric estimation.

Recall that Holder spaces contain smooth functions that are locally well approximated with polynomials or piecewise polynomial functions. Holder spaces are quite large and contain many interesting signals. However, Holder spaces are still inadequate in many applications. Often, we encounter functions that are not smooth everywhere; they contain discontinuities, jumps, spikes, etc. Indeed, the "singularities" (or non-smooth points) can be the most interesting and informative aspects of the functions. In Figures 1 and 2 we see examples of functions that are not-smooth everywhere, and furthermore have different degrees of smoothness in different regions.

2 NonLinear Approximation via Trees

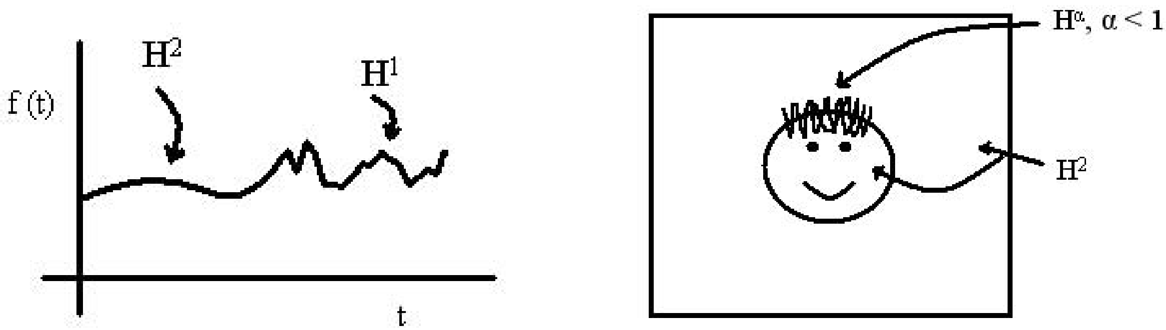
Let $B^\alpha(C)$ denote the set of all functions that are $H^\alpha(C)$ everywhere except on a set of measure zero (see example on Figure 3). To simplify the notation, we won't explicitly identify the domain (e.g., $[0,1]$ or $[0,1]^d$); that will be clear from the context.

Let's consider a 1-D case first. Let $f \in B^\alpha(C_\alpha)$ and consider approximating f by a piecewise polynomial function on a uniform partition.



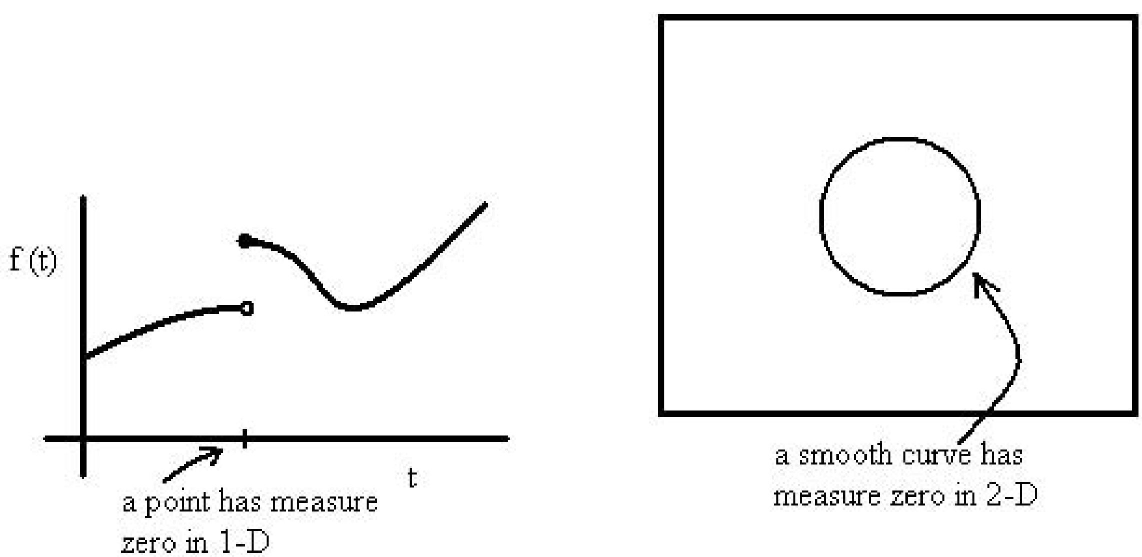
(a) (b)

Figure 1: Example of functions not smooth everywhere. (a) 1-D Case (b) 2-D Case



(a) (b)

Figure 2: Example of functions having different degrees of smoothness. (a) 1-D Case (b) 2-D Case



(a) (b)

Figure 3: Sets of measure zero. (a) 1-D Case (b) 2-D Case

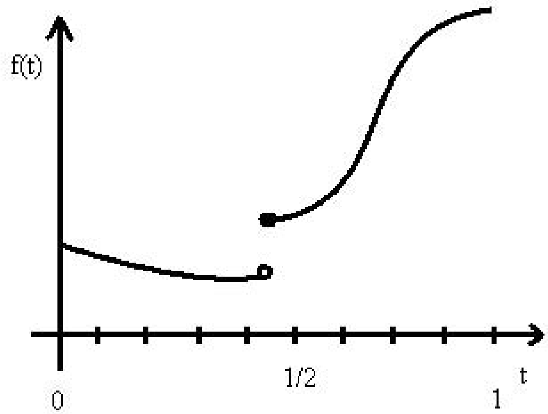


Figure 4: Smooth curve with a discontinuity.

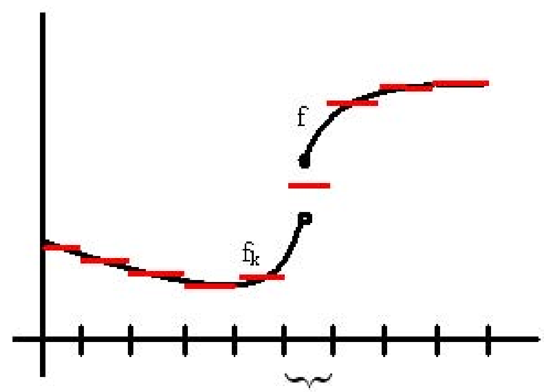


Figure 5: Large approximation error.

If f is Holder-α smooth everywhere, then by using an appropriate partition width k^{-1} and fitting degree $\lceil\alpha\rceil$ polynomials on each interval we have an approximation f_k satisfying

$$|f(t) - f_k(t)| \leq Ck^{-\alpha}$$

and

$$||f - f_k||_{L_2}^2 = O(k^{-2\alpha}) \ .$$

However, if there is a discontinuity then for t in the interval containing the discontinuity the difference

$$|f(t) - f_k(t)|$$

will not be small (see Figure 2). Suppose f is piecewise Lipschitz and f_k is a piecewise constant function. Then

$$|f(t) - f_k(t)| \approx \Delta$$

where Δ is a constant equal to average of f on right and left side of discontinuity in this interval (see Figure 2. Because of this

$$\Rightarrow ||f - f_k||_{L_2}^2 = O(k^{-1})$$

where k^{-1} is the width of the interval. Notice this rate is quite slower than if the function was globally smooth.

This problem naturally suggests the following remedy: adjust your partition so that the pieces are aligned with the discontinuities. To do so in a practical way implies that we want to use very small intervals near discontinuities and larger intervals in smooth regions. We can accomplish this need for "adaptive resolution" or "multiresolution" using recursive partitions and trees.

3 Recursive Dyadic Partitions

We discussed this idea already in our examination of classification trees. The main idea is illustrated in Figure 3

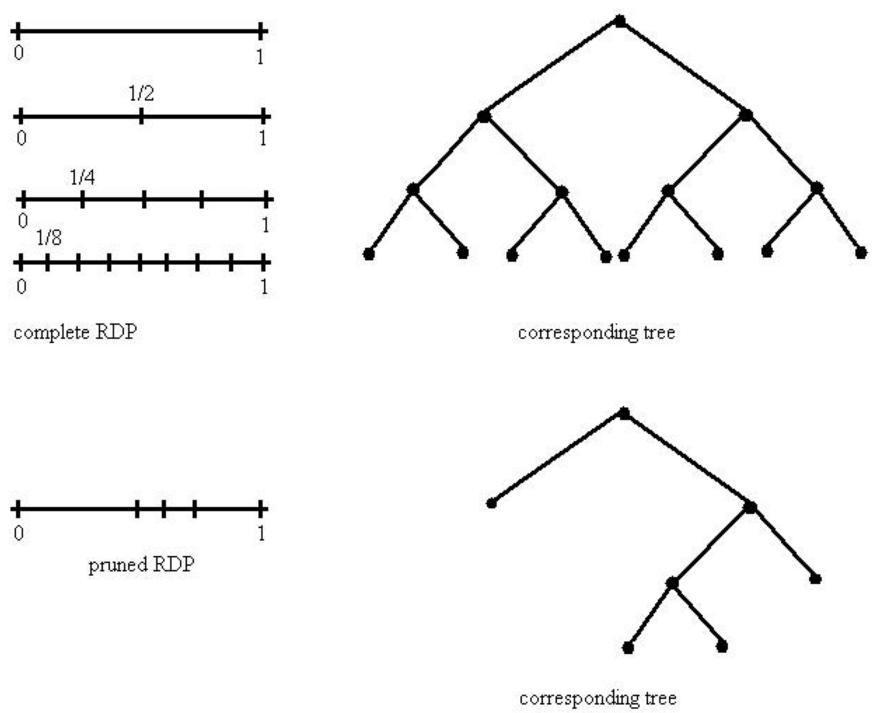


Figure 6: Complete and pruned RDP along with their correspnding tree structures.

Consider a function $f \in B^{\alpha}(C)$ that contains no more than m points of discontinuity, and is $H^{\alpha}(C)$ away from these points.

Lemma 1. *Consider a complete RDP with n intervals, then there exists an associated pruned RDP with* $O(klogn)$ *intervals, such that an associated piecewise degree* $\lceil\alpha\rceil$ *polynomial approximation* $\widetilde{(f)}_k$*, has a squared approximation error of* $O(min\{k^{-2\alpha}, n^{-1}\})$.

Proof : *Assume* $n > k > m$. *Divide* $[0,1]$ *into k intervals. If f is smooth on a particular interval I, then*

$$|f(t) - \tilde{f}_k(t)| = O(k^{-2\alpha}) \forall t \in I$$

.

In intervals that contain a discontinuity, recursively subdivide into two until the discontinuity is contained in an interval of width n^{-1}. This process results in at most $log_2 n$ additional subintervals per discontinuity, and the squared approximation error is $O(k{-}2\alpha)$ on all of them accept the m intervals of width n^{-1} containing the discontinuities where the error is $O(1)$ at each point.

Thus, the overall squared L_2 norm is

$$||f - \tilde{f}_k||^2_{L_2} = O(min\{k^{-2\alpha}, n^{-1}\})$$

and there are at most $k + log_2 n$ intervals in the partition. Since k¿m, we can upperbound the number of intervals by $2klog_2 n$.

Note that if the initial complete RDP has $n \approx k^{2\alpha}$ intervals, then the squared error is $O(k^{-2\alpha})$.

Thus, we only incur a factor of $2\alpha logk$ additional leafs and achieve the same overall approximation error as in the $H^\alpha(C_\alpha)$ case. We will see that this is a small price to pay in order to handle not only smooth functions, but also piecewise smooth functions.

4 Wavelet Approximations

Let $f \in L^2([0,1])$; $\int f^2(t)dt < \infty$.

A wavelet representation of f is a series of the form

$$f = c_o + \sum_{j \geq 0}^{\infty} \sum_{k=1}^{2^j} < f, \psi_{j,k} > \psi_{j,k} \ ,$$

where c_o is a constant ($c_o = \int_0^1 f(t)dt$),

$$\theta_{j,k} \equiv < f, \psi_{j,k} > = \int_0^1 f(t)\psi_{j,k}(t)dt$$

and the basis functions $\psi_{j,k}$ are orthonormal, oscillatory signals, each with an associated scale 2^{-j} and position $k2^{-j}$. $\psi_{j,k}$ is called the wavelet at scale 2^{-j} and position $k2^{-j}$.

Example 1. Haar Wavelets: *The "first" wavelet basis that was developed.*

$$\psi_{j,k}(t) = 2^{j/2}(\mathbf{1}\{t \in [2^{-j}(k-1), 2^{-j}(k-1/2)]\} - \mathbf{1}\{t \in [2^{-j}(k-1/2), 2^{-j}k]\})$$

$$\int_0^1 \psi_{j,k}(t)dt = 0$$

$$\int_0^1 \psi^2_{j,k}(t)dt = \int_{(k-1)2^{-j}}^{k2^{-j}} 2^j dt = 1$$

$$\int_0^1 \psi_{j,k}(t)\psi_{l,m}(t)dt = 0 \ unless \ j = l, k = m \ .$$

Note: *If f is constant on $[2^{-j}(k-1), 2^{-j}k]$, then*

$$\int f\psi_{j,k}(t) = 0$$

. Because of this and the fact that each wavelet basis function is supported on a small region means that wavelets are "blind" to constant patches.

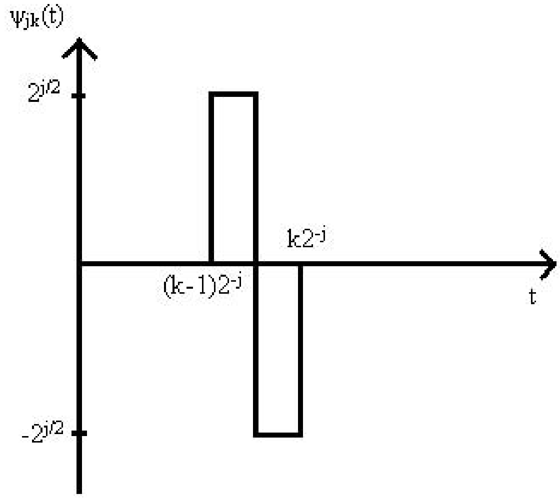


Figure 7: Haar Wavelet

Why is this basis useful? Suppose f is piecewise constant with at most m discontinuities. Let

$$f_J = c_o + \sum_{j=0}^{J-1} \sum_{k=1}^{2^j} \theta_{j,k} \psi_{j,k} \ ,$$

where $\theta_{j,k}$ are the wavelet coefficients. Then, f_J has at most mJ non-zero wavelet coefficients; i.e., $< f, \psi_{j,k} >= 0$ for all but mJ terms, since at most one Haar Wavelet at each scale senses each point of discontinuity. Said another way, all but at most m of the wavelets at each scale have support over constant regions of f. f_J itself will be piecewise constant with discontinuities only possible occurring at end points of the intervals $[2^{-J}(k-1), 2^{-J}k]$. Therefore, in this case

$$||f - f_J||_{L_2}^2 = O(2^{-J}).$$

The idea of Haar wavelets can be extended (although this extension is not immediately trivial). Daubechies wavelets are the extension of the Haar wavelet idea. Haar wavelets have one "vanishing moment":

$$\int_0^1 \psi_{j,k} = 0.$$

Daubechies wavelets are "smoother" basis functions with extra vanishing moments. The Daubechies-N wavelet has N vanishing moments.

$$\int_0^1 t^l \psi_{j,k} dt = 0, \quad \forall l = 0, 1, ..., N-1.$$

The Daubechies-1 wavelet is just the Haar case.

If f is a piecewise degree $\leq N$ polynomial with at most m pieces, then using the Daubechies-N wavelet system yields an approximation such that

$$||f - f_J||_{L_2}^2 = O(2^{-J})$$

and

$$f_J(t) = c_o + \sum_{j=0}^{J-1} \sum_{k=1}^{2^j} < f, \psi_{j,k} > \psi_{j,k}(t)$$

has at most $O(mJ)$ non-zero wavelet coefficients. f_J is called the **Discrete Wavelet Transform (DWT)** approximation of f. The key idea is the same as we saw with trees.

5 Sampled Data

We can also use DWT's to analyze and represent discrete, sampled functions. Suppose,

$$\boldsymbol{f} = [f(1/n), f(2/n), ..., f(n/n)]$$

then we can write $\boldsymbol{f}$ as

$$\boldsymbol{f} = c_o + \sum_{j=0}^{log_2 n-1} \sum_{k=1}^{2^j} < \boldsymbol{f}, \boldsymbol{\psi}_{j,k} > \boldsymbol{\psi}_{j,k}$$

where

$$\boldsymbol{\psi}_{j,k} = [\psi_{j,k}(1), \psi_{j,k}(2), ..., \psi_{j,k}(n)]$$

is a discrete time analog of the continuous time wavelets we considered before. In particular,

$$\sum_{i=1}^{n} i^l \psi_{j,k}(i) = 0, l = 0, 1, ..., N-1$$

for the Daubechies-N discrete wavelets.

$$< \boldsymbol{f}, \boldsymbol{\psi}_{j,k} >= \boldsymbol{f}^T \boldsymbol{\psi}_{j,k}$$

Thus, we also have an analogous approximation result: If $\boldsymbol{f}$ are samples from a piecewise degree $\leq N$ polynomial function with a finite number m of discontinuities, then $\boldsymbol{f}$ has $O(mJ)$ non-zero wavelet coefficients.

6 Approximating B^α functions with wavelets

Suppose $f \in B^\alpha(C)$ and has a finite number of discontinuities. Let f_p denote piecewise degree-$N(N = \lceil \alpha \rceil)$ polynomial approximation to f with $O(k)$ pieces; a uniform partition into k equal length intervals followed by addition splits at the points of discontinuity.

Then

$$|f(t) - f_p(t)|^2 = O(k^{-2\alpha}) \quad \forall t \in [0, 1]$$
$$\Rightarrow |f(i/n) - f_p(i/n)|^2 = O(k^{-2\alpha}), \quad i = 1, ..., n$$
$$\Rightarrow 1/n||\boldsymbol{f} - \boldsymbol{f}_p||_{\ell_2}^2 = O(k^{-2\alpha}))$$

and $\boldsymbol{f}_p$ has $O(klog_2 n)$ non-zero coefficients according to our previous analysis.

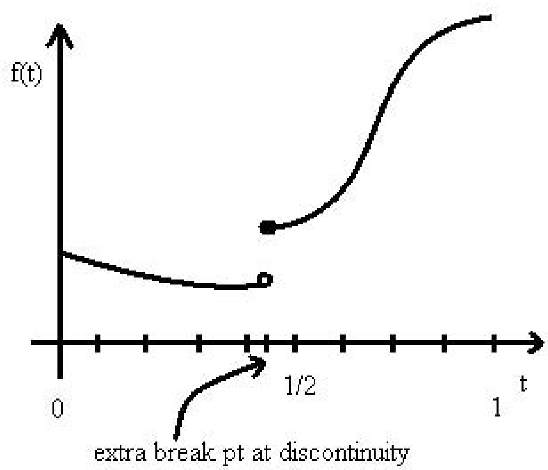


Figure 8: