ECE 901
Homework 10

Classification in Threshold Classes. Consider a binary classification problem where the feature space is $\mathcal{X} = [0,1]$, and the Bayes classifier has the form $f^*(x) = \mathbf{1}_{\{x \geq t^*\}}$ for a threshold $t^* \in (0,1)$.

1. In Homework 7 (question 1b) we considered two extra conditions:

i. P_X, the distribution of features, has a density function p_X that is bounded below by a constant $b > 0$.

ii. The conditional probability $\eta(x) := P(Y=1|X=x)$ satisfies $|\eta(x) - 1/2| \geq c$, for some constant $c > 0$.

Under these assumptions you showed that there exists a classifier whose excess risk is $O(\log n/n)$.

a. Prove that this is essentially the best you can do using a minimax approach. **HINT:** You can reduce the problem to a binary hypotheses testing, where the two hypotheses differ only on a small subset of the feature space $\mathcal{X}$.

b. Alternatively, instead of using a minimax approach, argue that this is essentially the best you can do by simply considering the bias/approximation error in the noiseless setting; i.e., when $\eta(x)$ is binary-valued.

2. In class we showed that if no conditions on η are made (in particular (ii) does not hold) then the best excess risk decay rate is $\sim \sqrt{1/n}$. We can envision scenarios in between those two extremes, in particular scenarios where η might "cross" the level 1/2, but not arbitrarily slowly. Let $f^*(x) = \mathbf{1}_{\{x \geq t^*\}}$ and suppose η is such that

$$|\eta(x) - 1/2| \geq c|x - t^*| \ , \tag{1}$$

for all x such that $|\eta(x) - 1/2| < \epsilon$, with $\epsilon > 0$. This implies that η varies *at least* linearly near t^*. Under this assumption, the problem is "easier" than the general problem (without an assumption on the variation of η), but allows for more difficult cases than the "jump" case above. Thus, we expect that one might be able to achieve an excess risk decay rate that is faster than $n^{-1/2}$, but slower than n^{-1}, when η satisfies (1).

Using the minimax technique, show that the expected excess risk cannot decay faster than $\sim n^{-2/3}$.

HINT: As in question 1, you can reduce the problem to a binary hypotheses test. Also note that condition (1) does not prevent η from "jumping" across 1/2 so you can use a slight modification of the construction you used for question 1.