

# ECE 901

## Homework 1

Suppose that we are interested in denoising an image. Define an image to be a mapping  $f : [0, 1]^2 \rightarrow R$ . Consider the observation model:

$$Y_{i,j} = f^*(i/n, j/n) + W_{i,j}, \quad i, j = 1, \dots, n$$

where  $\{W_{i,j}\}$  are zero-mean, finite variance noise terms. Given  $n^2$  noisy samples  $\{Y_{i,j}\}$  we wish to estimate  $f^*$ . For example, the image below was acquired by a digital camera. The noise is due to the low-light condition (limiting the number of photons hitting the CCD detector) in which the image was acquired. The image can be downloaded from [www.ece.wisc.edu/~nowak/SLT07](http://www.ece.wisc.edu/~nowak/SLT07).



1. Assume that  $f^*$  satisfies the 2-d Lipschitz condition:

$$|f^*(s_1, t_1) - f^*(s_2, t_2)| \leq L \max\{|s_1 - s_2|, |t_1 - t_2|\}.$$

Propose a sieve method for estimating  $f^*$  from the noisy data. Show that the estimator is consistent, and determine its rate of convergence (as a function of the number of samples  $n^2$ ).

2. Implement and test your estimator on the image above. Would you recommend this estimator? Why or why not?
3. Let's expand the space of possible functions from what we discussed in lecture. Consider the space of all 1-d *piecewise* Lipschitz smooth functions. An example of such a function is  $f(t) = f_1(t)1_{t \in [0, t_0]} + f_2(t)1_{t \in (t_0, 1]}$ , where  $f_1$  and  $f_2$  are Lipschitz and  $t_0 \in (0, 1)$ . Show that the sieve method discussed in class provides a consistent estimator for this (larger) class of functions and determine the optimal rate of convergence in this class.
4. Now extend the results from the case above to 2-d. Consider the space of all 2-d piecewise Lipschitz functions. This space includes images composed of Lipschitz smooth regions separated by boundaries (1-d curves). Show that the 2-d sieve method you developed above can produce a consistent estimator in this case and determine the optimal rate of convergence. Hint: If we have a partition of  $m$  2-d boxes, then a 1-d curve passes through  $O(\sqrt{m})$  of the boxes.