Daubechies' Wavelets

Theorem

The DTFT of $h(n)$ that has $R \leq \frac{N}{2}$ zeros at $\omega = \pi$ is

$$H(\omega) = \left( \frac{1 + e^{i\omega}}{2} \right)^K \mathbf{L}(\omega),$$

satisfies

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$$

if and only if $L(\omega) = |L(\omega)|^2$ can be written as

$$L(\omega) = P \left( \sin^2 \left( \frac{\omega}{2} \right) \right)$$

where $P(\cdot)$ is a polynomial of the form

$$P(y) = \sum_{k=0}^{K-1} \left( \frac{K-1+k}{k} \right) y^k + y^K R \left( \frac{1}{2} - y \right)$$

and $R(\cdot)$ is an odd polynomial chosen so that $P(y) \geq 0$ for $0 \leq y \leq 1$. 
This theorem shows that a $K$-regular scaling filter (i.e., a scaling filter associated with a wavelet with $K$ vanishing moments) satisfies

$$|H(\omega)|^2 = \left| \frac{1+e^{i\omega}}{2} \right|^{2K} P\left( \sin^2 \left( \frac{\omega}{2} \right) \right)$$

To find $H(\omega)$, and hence $h(n)$, we must find the square root of this equation. This is known as spectral factorization. In general, this is carried out numerically, but in some cases it turns out that several factorizations will work. That is, this does not give a unique $h(n)$. The Daubechies' filters we looked at earlier are the solutions with minimum phase.
Ex. D4 scaling filter

\[
\begin{align*}
    h(0) &= \left(1 - \cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right) / (2\sqrt{2}) = 0.48296 \\
    h(1) &= \left(1 + \cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right) / (2\sqrt{2}) = 0.83652 \\
    h(2) &= \left(1 + \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right) / (2\sqrt{2}) = 0.22414 \\
    h(3) &= \left(1 - \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right) / (2\sqrt{2}) = -0.12941 \\
\end{align*}
\]

**Vanishing moments** \( k = N/2 = 2 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \mu_k )</th>
<th>( \nu_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.2247</td>
<td>0.2165</td>
</tr>
<tr>
<td>3</td>
<td>0.5720</td>
<td>0.7868</td>
</tr>
</tbody>
</table>

Wavelet filter: \( h(n) \)
Wavelet func: \( \psi(t) \)

Zeros of \( H(z) = \sum_{n=0}^{N} h(n) z^{-n} \)

\( z = -1, -1, 0.2679 \)
**Ex. D6 scaling filter**

<table>
<thead>
<tr>
<th>n</th>
<th>h(n)</th>
<th>$h_1(n) = (-1)^{n+1} h(5-n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3327</td>
<td>-0.0352</td>
</tr>
<tr>
<td>1</td>
<td>0.8069</td>
<td>-0.0854</td>
</tr>
<tr>
<td>2</td>
<td>0.4599</td>
<td>0.1350</td>
</tr>
<tr>
<td>3</td>
<td>-0.1350</td>
<td>0.4599</td>
</tr>
<tr>
<td>4</td>
<td>-0.0854</td>
<td>-0.8069</td>
</tr>
<tr>
<td>5</td>
<td>0.0352</td>
<td>0.3327</td>
</tr>
</tbody>
</table>

**Vanishing Moments**  \[ k^* = \frac{N}{2} = 3 \]

<table>
<thead>
<tr>
<th>k</th>
<th>$\alpha_1(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3.3541</td>
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<tr>
<td>4</td>
<td>40.6797</td>
</tr>
<tr>
<td>5</td>
<td>329.3237</td>
</tr>
</tbody>
</table>

$\{ \text{1st three moments are zero} \}$  \[ \Rightarrow \text{wavelets are "blind" to quadratic polynomials} \]

**Zeros of H(Z)**

\[ z = -1, -1, -1, \; 0.2873 \pm 0.1529i \]
\[ \alpha = 1.3598 \ldots \]
\[ \beta = -0.7821 \ldots \]
$\alpha = \frac{\pi}{3}$

$\beta = -\frac{\pi}{3}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\mu_1(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.0860</td>
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<tr>
<td>2</td>
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<td>4</td>
<td>18.3214</td>
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<tr>
<td>5</td>
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scaling filter $h(n)$

magnitude of DTFT of $h(n)$

scaling function

associated wavelet
Daubechies' Method For Wavelet Design

We seek a scaling filter \( h(n) \) satisfying

1. \( \sum_n h(n) = \sqrt{2} \)

2. \( \sum_n h(n) h(n-2k) = \delta(k) \)

3. \( \sum_n n^k h_1(n) = 0 \), \( k = 0, 1, \ldots, K-1 \)

or equivalently

\( \sum_n n^k (-1)^n h(1-n) = 0 \)

Note 1. and 3. are linear equations
in \( h(n) \), 2. is quadratic.

Also, in order to have \( K \) vanishing
moments, \( h(n) \) must be of length \( 2K \)
or more.

Assuming \( h(n) \) is length \( N = 2K \),
we have precisely \( K \) linear equations,
\( K \) quadratic equations, and \( 2K \)
unknowns. How do we solve this
system of nonlinear equations?
Recall that 3. (Vanishing moments) is equivalent to \( \mathbb{R} \) regularity of \( h(n) \):

\[
H(\omega) = \left( \frac{1 + e^{i\omega}}{2} \right) \mathbb{R} \ L(\omega)
\]

This implies that

\[
M(\omega) \equiv |H(\omega)|^2 = \left| \frac{1 + e^{i\omega}}{2} \right|^2 \ L(\omega)
\]

where

\[
L(\omega) \equiv |L(\omega)|^2
\]

Note

\[
\left| \frac{1 + e^{i\omega}}{2} \right|^2 = \left( \frac{1 + e^{i\omega}}{2} \right) \left( \frac{1 + e^{-i\omega}}{2} \right) = \frac{1 + \cos(\omega)}{2}
\]

\[
= \cos^2 \left( \frac{\omega}{2} \right)
\]

So

\[
M(\omega) = \left| \cos^2 \left( \frac{\omega}{2} \right) \right| \ L(\omega)
\]

Also,

\[
|H(\omega)|^2 = H(\omega) H^*(\omega) = H(\omega) H(-\omega)
\]

(since \( h(n) \) is real-valued)

\[
\Rightarrow |H(\omega)|^2 = |H(-\omega)|^2
\]

\[
\Rightarrow |H(\omega)|^2 \text{ is even fnc of } \omega.
\]
We have

\[ M(w) = \left| \cos^2 \left( \frac{w}{2} \right) \right|^K L(w) \]

if even \quad \text{if even} \quad \text{if even}

So \( L(w) \) is a polynomial in \( \cos(w) \) (instead of \( e^{i\omega} = \cos(w) + i\sin(w) \))

It is convenient to express \( L(w) \)

in terms of \( \sin^2 \left( \frac{w}{2} \right) = \frac{1 - \cos(w)}{2} \) instead:

\[ L(w) = p\left( \sin^2 \left( \frac{w}{2} \right) \right) \quad \text{degree } K-1 \]

Now we have

\[ M(w) = \left| \cos^2 \left( \frac{w}{2} \right) \right|^K p \left( \sin^2 \left( \frac{w}{2} \right) \right) \]

Recall that \( \mathbf{a} \), (quadratic constraints)

\[ M(w) + M(w+\pi) = 1 \quad \text{ (see p. 148)} \]

In terms of \( P \), thus becomes

\[ \left| \cos^2 \left( \frac{w}{2} \right) \right|^K P \left( \sin^2 \left( \frac{w}{2} \right) \right) + \left| \sin^2 \left( \frac{w}{2} \right) \right|^K P \left( \cos^2 \left( \frac{w}{2} \right) \right) = \]
Let $y = \cos^2(\frac{\theta}{2})$. Then we have

$$y^K P(1-y) + (1-y)^K P(y) = 1$$

for all $y \in [0,1]$. Since $P$ is a polynomial, this implies that we have equality for all $y \in \mathbb{R}$.

To find $P$ satisfying this equation we need Bezout's theorem.

**Theorem:** If $p_1$ and $p_2$ are polynomials of degree $n_1$ and $n_2$, respectively, with no common zeros, then there exist unique polynomials $q_1$ and $q_2$ of degree $n_2-1$ and $n_1-1$, respectively, such that

$$p_1(x)q_1(x) + p_2(x)q_2(x) = 1$$

**Proof:** Straightforward inductive argument.
Bezout's theorem shows that

\[ y^K p(1-y) + (1-y)^K p(y) = 1 \]

has a unique solution \( p(y) \),
which is a polynomial of
degree \( \leq K-1 \).

Re-arranging this expression we have

\[ p(y) = (1 - y^K p(1-y)) (1 - y)^{-K} \]

The Taylor's series of \( (1-y)^{-K} \) is

\[ (1-y)^{-K} = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k + R_K(y) \]

where

\[ R_K(y) = \sum_{k \geq K} a_k y^k \]
So,

\[ P(y) = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k + R_K(y) y^K p(1-y) \]

\[ = \sum_{k \geq K} a_k y^k \]

We know that degree \( P(y) \leq K-1 \), hence

\[ P(y) = \sum_{k=0}^{K-1} \binom{K+k-1}{k} y^k \]

Thus, we now have

\[ |H(\omega)|^2 = \left| \cos^2 \left( \frac{\omega}{2} \right) \right|^K P \left( \sin^2 \left( \frac{\omega}{2} \right) \right) \]

\[ = \left| \left( \frac{1+e^{j\omega}}{2} \right) \left( \frac{1-e^{j\omega}}{2} \right) \right|^K P \left( \frac{1}{2} - \frac{e^{j\omega}}{2} - \frac{e^{-j\omega}}{2} \right) \]

or with \( z = e^{j\omega} \)

\[ |H(z)|^2 = \left| \left( \frac{1+z}{2} \right) \left( \frac{1-z}{2} \right) \right|^K P \left( \frac{1}{2} - \frac{z}{2} - \frac{z^{-1}}{2} \right) \]
\[ |H(z)|^2 \] is simply a polynomial in \( z \). We can factor it using the "roots" command in Matlab.

If we choose the \( N-1 \) roots of smallest magnitude, then we get \( N-1 \) roots corresponding to a minimum phase factorization of \( H(z) \). If \( z_0, \ldots, z_{N-1} \) are these roots, then

\[
H(z) = \prod_{n=0}^{N-1} \left( z^{-1} z_n \right)
\]

\[
= \sum_{n=0}^{N-1} h(n) z^{-n}
\]

and hence we can identify \( h(n) \). This is the standard Daubechies length \( N \) filter.