Perfect Reconstruction Filter Banks and Multiresolution Analysis

In this lecture we will:

- determine conditions that guarantee the filter bank implementation of the DWT is invertible

- relate the wavelet filter to the scaling filter

- establish necessary and sufficient conditions for the scaling filter that guarantee a valid MRA.

- relate scaling and wavelet filters to the corresponding continuous scaling and wavelet functions
Perfect Reconstruction Filter Banks

Consider a simple, two-channel filter bank:

\[ x(n) \xrightarrow{h, \downarrow 2} c(n) \xrightarrow{42} \hat{h} \xrightarrow{\downarrow 2} x'(n) \]

\[ x(n) \xrightarrow{g, \downarrow 2} d(n) \xrightarrow{42} \hat{g} \]

We want \( x'(n) = x(n) \) \underline{Perfect Reconstruction}

What conditions must the filters \((h, g, \hat{h}, \hat{g})\) satisfy to guarantee PR?

\[ \hat{h} = h \underline{scaling \ filter} \]
\[ g = \hat{g} = h, \underline{wavelet \ filter} \]

for DWT

\[ x(n) \xrightarrow{h} y(n) \]
\[ y(n) = x * h(-n) \] \( \text{(time-reversed filter)} \)

Note:
Theorem 1 (Vetterli):

The filterbank performs an exact reconstruction for every input signal if and only if

\[ H^*(\omega + \pi) \tilde{A}(\omega) + G^*(\omega + \pi) \tilde{G}(\omega) = 0 \]

and

\[ H^*(\omega) \tilde{A}(\omega) + G^*(\omega) \tilde{G}(\omega) = 2 \]

proof:

First consider

\[ X(n) \rightarrow \text{filter} \rightarrow \frac{1}{2} \rightarrow C(n) \]

The Fourier transform of \( C(n) \) can be written as

\[ C(2\omega) = \sum_n c'(2n) e^{-j2\omega n} \]

or

\[ C(2\omega) = \sum_n c'(2n) e^{-j2(\omega + \pi)n} \]

\[ \text{since } e^{-j2\pi n} = 1 \]
So, we have
\[
C(2\omega) = \frac{1}{2} \left( \sum_n c'(zn)e^{-j2\omega n} + \sum_n c'(zn)e^{-j2(\omega+\pi)n} \right)
\]
\[
= \frac{1}{2} \left( C'(\omega) + C'(\omega+\pi) \right)
\]
\[
= \frac{1}{2} \left( X(\omega)H'^*(\omega) + X(\omega+\pi)H'^*(\omega+\pi) \right)
\]
\[\text{conjugation due to time reversal of filter}\]

Similarly,
\[
D(2\omega) = \frac{1}{2} \left( X(\omega)G'^*(\omega) + X(\omega+\pi)G'^*(\omega+\pi) \right)
\]

Next, we up-sample \(c(n)\) and \(d(n)\)

\[c''(n) = \begin{cases} 
  c(p), & n = 2p \\
  0, & \text{otherwise}
\end{cases}\]

\[d''(n) = \begin{cases} 
  d(p), & n = 2p \\
  0, & \text{a.w.}
\end{cases}\]
Note that
\[
C''(w) = \sum_n c''(n) e^{-jwn}
\]
\[
= \sum_n c(n) e^{-jwan}
\]
\[
= C(2w)
\]

Similarly,
\[
D''(w) = D(2w).
\]

Finally, the up-sampled signals \(c''(n)\) and \(d''(n)\) are filtered by \(\tilde{h}\) and \(\tilde{g}\), resp. Thus, the output \(X'(w)\) has DTFT

\[
X'(w) = C(2w) \tilde{H}(w) + D(2w) \tilde{G}(w)
\]
\[
= \frac{1}{2} \left[ X(w) H^*(w) + X(w+\pi) H^*(w+\pi) \right] \tilde{H}(w)
\]
\[
+ \frac{1}{2} \left[ X(w) G^*(w) + X(w+\pi) G^*(w+\pi) \right] \tilde{G}(w)
\]
Re-arranging terms,

\[ X'(w) = \frac{1}{2} \left[ H^*(w) \hat{H}(w) + G^*(w) \hat{G}(w) \right] X(w) \]

\[ + \frac{1}{2} \left[ H^*(w+\pi) \hat{H}(w) + G^*(w+\pi) \hat{G}(w) \right] X(w+\pi) \]

We want \( X'(w) = X(w) \), so we require

\[ H^*(w) \hat{H}(w) + G^*(w) \hat{G}(w) = 2 \]

and

\[ H^*(w+\pi) \hat{H}(w) + G^*(w+\pi) \hat{G}(w) = 0 \]
In fact, the previous theorem shows that the synthesis filters \( \hat{h} \) and \( \hat{g} \) are determined by the analysis filters \( h \) and \( g \), as shown by the following argument.

In matrix form, the PR conditions become,

\[
\begin{bmatrix}
    H(w) & G(w) \\
    H(w+\pi) & G(w+\pi)
\end{bmatrix}
\begin{bmatrix}
    \hat{H}^*(w) \\
    \hat{G}^*(w)
\end{bmatrix}
= 
\begin{bmatrix}
    2 \\
    0
\end{bmatrix}
\]

Inverting this equation gives

\[
\begin{bmatrix}
    \hat{H}^*(w) \\
    \hat{G}^*(w)
\end{bmatrix}
= \frac{2}{\Delta(w)}
\begin{bmatrix}
    G(w+\pi) \\
    -H(w+\pi)
\end{bmatrix}
\]

where \( \Delta(w) \) is the determinant

\[
\Delta(w) = H(w) G(w+\pi) - H(w+\pi) G(w)
\]
When all filters are FIR, 
$\Delta(w)$ can be evaluated directly, 
and simpler relations between the 
analysis and synthesis filters can be 
obtained.

**Theorem 2** (p. 262 Mallat)

Perfect reconstruction filters satisfy

$$H^*(w) \hat{H}(w) + H^*(w+\pi) \hat{H}(w+\pi) = 2.$$ \(\Box\)

For FIR filters, there exists $\alpha \in \mathbb{R}$ and 
$l \in \mathbb{Z}$ such that

$$G(w) = \alpha e^{-i(2l+1)w} H^*(w+\pi)$$

$$\hat{G}(w) = \alpha^{-1} e^{-i(2l+1)w} H^*(w+\pi).$$
**Proof:**

The matrix inverse equation shows that

\[ \tilde{\mathbf{H}}^*(\omega) = \frac{2}{\Delta(\omega)} \mathbf{G}(\omega + \pi) \]

\[ \tilde{\mathbf{G}}^*(\omega) = \frac{-2}{\Delta(\omega)} \mathbf{H}(\omega + \pi) \]

Therefore,

\[ \mathbf{G}(\omega) \tilde{\mathbf{G}}^*(\omega) = -\frac{\Delta(\omega + \pi)}{\Delta(\omega)} \tilde{\mathbf{H}}^*(\omega + \pi) \mathbf{H}(\omega + \pi) \]

Note that

\[ \Delta(\omega + \pi) = \mathbf{H}(\omega \pi) \mathbf{G}(\omega) - \mathbf{H}(\omega) \mathbf{G}(\omega + \pi) \]

since \( \mathbf{H}, \mathbf{G} \) are 2\( \pi \)-periodic

\[ = -\Delta(\omega) \]

\[ \Rightarrow \]

\[ \mathbf{G}(\omega) \tilde{\mathbf{G}}^*(\omega) = \tilde{\mathbf{H}}^*(\omega + \pi) \mathbf{H}(\omega + \pi) \]

From Thm 1, \( \mathbf{H}^*(\omega) \tilde{\mathbf{H}}(\omega) + \mathbf{G}^*(\omega) \tilde{\mathbf{G}}(\omega) = 2 \)

\[ \Rightarrow \mathbf{H}^*(\omega) \tilde{\mathbf{H}}(\omega) + \tilde{\mathbf{H}}^*(\omega + \pi) \mathbf{H}(\omega + \pi) = 2 \].
Now, to obtain a relation between \( g, \tilde{g} \) and \( h, \tilde{h} \) we note the following.

(i) the DTFT of an FIR filter is a finite series in \( e^{j \omega} \)

(ii) if all filters are FIR, then \( \Delta(w) \) is a finite series

(iii) since \( \tilde{H}(w) \) and \( \tilde{G}(w) \) must be finite series, the facts

\[
\tilde{H}^*(w) = \frac{2}{\Delta(w)} G(w+\pi), \quad \tilde{G}^*(w) = -\frac{2}{\Delta(w)} H(w+\pi)
\]

imply that

\[
\frac{1}{\Delta(w)} \text{ is also a finite series.}
\]

However, the only DTFT whose inverse and itself are both finite series is a one-term series:

\[
\Rightarrow \quad \Delta(w) = \beta e^{j \omega m}
\]

\[
\frac{1}{\Delta(w)} = \frac{1}{\beta} e^{-j \omega m}
\]
Furthermore, since
\[ \Delta(w) = -\Delta(w + \pi) \]
the exponent \( m \) must be odd.

Thus,
\[ \Delta(w) = \beta e^{i(2k+1)w} \]

\[ \Rightarrow \quad G(w) = \frac{\beta e^{i(2k+1)w}}{a} \tilde{H}(w + \pi) \]
\[ \tilde{G}(w) = -\frac{2}{\beta e^{i(2k+1)w}} H^*(w + \pi) \]
\[ = -\left(\frac{2}{\beta}\right) e^{i(2k+1)w} H^*(w + \pi) \]

Take \( \alpha = \frac{\beta}{2} \) to finish the proof.
The values \( \alpha \in \mathbb{R} \) and \( l \in \mathbb{Z} \) are arbitrary. In particular, with \( \alpha = 1 \) and \( l = 0 \) we obtain

\[
G(w) = e^{-iw} H^*(w+\pi)
\]

\[
\leftarrow \quad g(n) = (-1)^{1-n} h(1-n)
\]

\[
\tilde{G}(w) = e^{-iw} H^*(w+\pi)
\]

\[
\leftarrow \quad \tilde{g}(n) = (-1)^{1-n} h(1-n)
\]

**Special Case:** DWT \( h = \tilde{h} \)

**Smith-Barnwell Condition**

\[
|H(w)|^2 + |H(w+\pi)|^2 = 2
\]

(Quadrature mirror filter)

\[
g(n) = (-1)^{1-n} h(1-n)
\]
Filter Banks and DWT

\[
g(n) = (-1)^{(1-n)} h(1-n)
\]

\[
|H(w)|^2 + |H(w+\pi)|^2 = 2
\]

(Pr condition)

Does the iterated filter bank produce a valid MRA?
Theorem 3  \hspace{1cm} \text{(Time-domain PR condition)}

For any FIR \( h(n) \)

\[
|H(w)|^2 + |H(w + \pi)|^2 = 2
\]

\[\iff \sum_{n} h(n) h(n-2k) = \delta(k)\]

**Proof:**

Let \( Y(k) = \sum_{n} h(n) h(n-2k) \)

\[
Y(2w) = \sum_{k} \left( \sum_{n} h(n) h(n-2k) \right) e^{-j 2w k}
\]

\[
= \sum_{n} h(n) \sum_{k} h(n-2k) e^{-j w 2k}
\]

\[
= \sum_{n} h(n) \sum_{m} h(m) e^{-j w (n-m)}
\]

\[
= H(w) \cdot H^*(w)
\]

\[
= |H(w)|^2
\]
Also, since DTFTs are $2\pi$-periodic

$$Y(2w) = Y(2(w+\pi))$$

$$= |H(w+\pi)|^2$$

So, we can write

$$Y(2w) = \frac{1}{2} \left( |H(w)|^2 + |H(w+\pi)|^2 \right)$$

$$= 1 \quad \text{(according to PR condition)}$$

$$\Rightarrow \quad y(n) = \delta(n)$$

Alternatively,

if \hspace{1cm} \sum h(n) h(n-2k) = \delta(k),

then

$$|H(w)|^2 + |H(w+\pi)|^2 = 2.$$
Perfect Reconstruction and MRA

It turns out that the PR condition

\[ |H(w)|^2 + |H(w+\pi)|^2 = 2 \]

or equivalently

\[ \sum_{n} h(n) h(n-2k) = \delta(k) \]

is also key to establishing that \( h(n) \) generates a valid MRA and an orthonormal wavelet basis for \( L^2(\mathbb{R}) \).
Theorem 4 (Cohen)

Let \( h(n) \) be an FIR scaling filter and let \( H(\omega) \) be its DTFT.

If

(i) \[ \sum h(n) = \sqrt{2} \]

(ii) \[ \sum h(n) h(n-2k) = \delta(k) \]

(iii) \( H(\omega) \neq 0 \) for \( -\frac{\pi}{3} \leq \omega \leq \frac{\pi}{3} \)

Then the scaling equation

\[ \phi(t) = \sum h(n) \sqrt{2} \phi(2t-n) \]

has a unique solution \( \phi \in L^2(\mathbb{R}) \) and \( \{ \phi(t-k) \} \) are orthogonal.

Moreover, (i), (ii), and (iii) guarantee that \( h(n) \) produces an orthogonal MRA of \( L^2(\mathbb{R}) \).
proof:

To show that (i), (ii), and (iii) are sufficient to guarantee an orthogonal MRA takes a lot of work (see A. Cohen's book).

So, here we will just establish that (i) and (ii) are necessary conditions. This illustrates the importance of these two key properties of the scaling filter.

Necessity of (i):

Let \( \phi \) be a solution to the scaling equation. Then

\[
\phi(t) = \sum_n h(n) \sqrt{2} \phi(2t-n)
\]

\[\Rightarrow \int_{-\infty}^{\infty} \phi(t) dt = \int_{-\infty}^{\infty} \sum_n h(n) \sqrt{2} \phi(2t-n) dt\]
Interchanging summation and integration

\[
\int_{-\infty}^{\infty} \phi(t) dt = \sum_{n} h(n) \sqrt{2} \int_{-\infty}^{\infty} \phi(zt-n) dt
\]

\[
= \sum_{n} h(n) \sqrt{2} \int_{-\infty}^{\infty} \phi(x) \frac{1}{z} dx, \quad x=zt
\]

\[\Rightarrow \quad 1 = \frac{1}{\sqrt{2}} \sum_{n} h(n)\]

or \[\sum_{n} h(n) = \sqrt{2}\]

Thus, if \( \phi \) is a solution of the scaling equation, \( h(n) \) must satisfy (i).

Necessity of (ii):

Let \( \phi \) be a solution of the scaling equation and furthermore assume that \( \{\phi(t-n)\} \) are orthonormal.
Orthogonality means

\[ \int \phi(t) \phi(t-k) \, dt = 0 \quad \text{for } k \neq 0 \quad \square \]

\[ \Rightarrow \quad \int \sum_n h(n) \sqrt{2} \phi(2t-n) \cdot \sum_n h(m) \sqrt{2} \phi(2t-2k-m) \, dt = 0 \quad \square \]

\[ \Rightarrow \quad \sum_m \sum_n h(n) h(m) \int \phi(x-2k-m) \phi(x-n) \, dx = 0 \quad \# \]

where \( x = 2t \).

\[ \square \] also implies

\[ \int \phi(x-2k-m) \phi(x-n) \, dx = \delta(n-2k-m) \]

So, \[ \square \] becomes

\[ \sum_n h(n) h(n-2k) = 0 \quad k \neq 0. \]
Summary:

(i) \[ \sum_{n} h(n) = \sqrt{2} \]

is necessary for any solution to the scaling equation

\[ \phi(t) = \sum_{n} h(n) \sqrt{2} \phi(2t-n) \]

(ii) \[ \sum h(n) h(n-2k) = \delta(k) \]

is necessary to ensure that \( \{ \phi(t-n) \} \) are orthonormal.

Also, this is equivalent to PR condition

\[ |H(\omega)|^2 + |H(\omega+i\pi)|^2 = 2 \]

(iii) \( H(\omega) \neq 0 \) on \( -\frac{\pi}{3} \leq \omega \leq \frac{\pi}{3} \)

is an extra requirement needed for proving that \( h(n) \) will generate a valid MRA.

It basically says that \( h(n) \) must be a lowpass filter.