Multidimensional Wavelet Representations

It is possible to construct wavelet representations and DWTs for multidimensional signals. In particular, we will be interested in 2-d DWTs for image analysis.

The basic scale space construction is completely analogous to the 1-d case.

We begin by specifying a scaling function \( \phi(x) \) where \( x \in \mathbb{R}^n \) and generate a space

\[
V_0 = \{ \phi(x - k) \}_{k \in \mathbb{Z}^n}
\]

\( n \)-tuples of integers
Ex. 2-d Haar scaling function

Let \( x = (x, y) \) point in the plane

\[
\Phi(x, y) = \begin{cases} 
1, & 0 \leq x < 1, 0 \leq y < 1 \\
0, & \text{o.w.}
\end{cases}
\]

This is a very natural approximation function for images since we normally associate each pixel value with the integral of the underlying (continuous) image intensity function over a small square region of space.
2-d MRA

The 2-d multiresolution subspaces are generated by translates and dilates of the scaling function.

\[ V_j = \{ a^{j(n_2)} \phi (2^j r - k) \}_{k \in \mathbb{Z}^n} \]

And we have

\[ \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(\mathbb{R}^n) \]

Ex. Haar scaling functions

\[ \phi(a^{-j}x-1, a^{-j}y-1) \]
2-d Wavelet Subspaces

Difficulty:

Going from scale $2^{-j}$ to $2^{-(j+1)}$ (equiv. resolution $2^j$ to $2^{j-1}$) involves a loss of information in the ratio of $2^n : 1$.

Ex. Haar Analysis

At scale $2^{-j}$, each $2^{-j+1} \times 2^{-j+1}$ region of the plane is "analyzed" by 4 scaling functions. In contrast, at scale $2^{-(j+1)}$, only one scaling function analyzes the same region.
To account for the $2^n : 1$ loss of information between $V_j$ and $V_{j-1}$, it is necessary to have $2^{n-1}$ wavelet functions (or equivalently $2^{n-1}$ highpass filters) to carry this lost info.

Ex. Haar Analysis

To represent the difference between $V_j$ and $V_{j-1}$, we need to represent the deviations of the $V_j$ piecewise constant approximation over the region $[0, 2^{-j_1}) \times [0, 2^{-j_1})$ from the $V_{j-1}$ constant approximation over the same region.
These deviations or "details" are represented by projecting the image onto the 2-d Haar wavelets:

\[ \psi_1(x, y) \]
\[ \psi_2(x, y) \]
\[ \psi_3(x, y) \]

Note:
- \( \psi_1 \) carries the horizontal deviation (i.e., senses vertical edges or details)
- \( \psi_2 \) carries vertical deviation (i.e., senses horizontal edges)
- \( \psi_3 \) carries diagonal deviation (i.e., sensitive to diagonal structure)
2-d Wavelet Subspaces

So, in 2-d we have 3 wavelet subspaces at each scale

\[ W_j^1, W_j^2, \text{ and } W_j^3 \]

\[ V_{j+1} = V_j \oplus W_j^1 \oplus W_j^2 \oplus W_j^3 \]

Ex. Let \( \psi_{j,m,n} = \psi^j(2^j x - m, 2^j y - n) \) (Haar wavelets)

Verify that

\[ P_{V_{j+1}} f(x,y) = P_{V_j} f(x,y) + P_{W_j^1} f(x,y) \]

\[ + P_{W_j^2} f(x,y) + P_{W_j^3} f(x,y) \]

Hint: It suffices to show that each scaling function \( \phi(2^{j+1} x + m, 2^{j+1} y + n) \)

can be expressed as a linear combination of \( \{ \phi(2^j x + m, 2^j y + n) \}_{j=1,2,3} \) and \( \{ \psi^j(2^j x + m, 2^j y + n) \}_{j=1,2,3} \)
Note that the Haar wavelets

\[ \psi'(x,y) \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
-\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]

\[ \psi^2(x,y) \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
-\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]

\[ \psi^3(x,y) \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
-\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]
\[ \begin{array}{|c|c|}
\hline
\frac{1}{2} & -\frac{1}{2} \\
\hline
\frac{1}{2} & \frac{1}{2} \\
\hline
\end{array} \]

can all be represented as products of one-dimensional Haar wavelets and scaling functions

\[ \psi'(x,y) = \psi(x) \phi(y) \]
\[ \psi^2(x,y) = \phi(x) \psi(y) \]
\[ \psi^3(x,y) = \psi(x) \psi(y) \]

where

\[ \phi(x) = \begin{cases} 
1, & 0 \leq x < 1 \\
0, & \text{otherwise}
\end{cases} \]

\[ \psi(x) = \begin{cases} 
1, & 0 \leq x < \frac{1}{2} \\
-1, & \frac{1}{2} \leq x < 1 \\
0, & \text{otherwise}
\end{cases} \]
A similar construction holds in general.

**Theorem:** Let \( \phi \) be a scaling function and \( \psi \) be the corresponding wavelet generating an orthogonal MRA of \( L^2(\mathbb{R}) \). Define three wavelets:

\[
\psi'(x,y) = \phi(x)\phi(y) \\
\psi^2(x,y) = \phi(x)\psi(y) \\
\psi^3(x,y) = \phi(x)\phi(y)
\]

and for \( l = 1, 2, 3 \) let

\[
\psi_{j,m,n}^l = 2^j \psi(2^j x - m, 2^j y - n).
\]

Then

\[
\{ \psi_{j,m,n}^l \}_{j,m,n \in \mathbb{Z}^2} \text{ is an o.n. basis for } W_j^l.
\]

and

\[
\{ \psi_{j,m,n}, \psi_{j,m,n}^2, \psi_{j,m,n}^3 \}_{j,m,n \in \mathbb{Z}^2} \text{ is an o.n. basis for } L^2(\mathbb{R}^2).
\]
The 2-d DWT

Let \( f(x, y) \) be a 2-d image, \( x, y \in \mathbb{R}^2 \), and let \( \{ \Phi(x-m, y-n) \}_{m,n} \) be an o.n. basis for \( V_0 \) and
\[
\{ \psi_{j,m,n}^l(x,y) \}_{m,n} \text{ be an o.n. basis for } W^l_j, \quad l = 1, 2, 3, \quad j \geq 0.
\]

Then we can write
\[
f(x, y) = \sum_{m,n=-\infty}^{\infty} c_0(m,n) \Phi(x-m, y-n) + \sum_{j \geq 0} \sum_{l=1}^{3} \sum_{m,n=-\infty}^{\infty} d^l_j(m,n) \psi_{j,m,n}^l(x,y)
\]

where
\[
c_0(m,n) = \langle f, \Phi(x-m, y-n) \rangle
\]
\[
d^l_j(m,n) = \langle f, 2^j \psi(2^j x-m, 2^j y-n) \rangle
\]
\[
\{ d^l_j(m,n) \}_{m,n} = \text{wavelet coefficients at scale } 2^{-j} \text{ and orientation } l.
\]
Computing the 2-d DWT

1. Assume an initial set of scaling coefficients \( \{ C_J(m, n) \} \) representing an approximation \( f_j = P_{V_j} f \) (\( \approx f \) if \( J \) is sufficiently large) to an image \( f \) at scale \( J \).

   In practice, \( \{ C_J(m, n) \} \) are the pixel values of a digital image.

2. The wavelet and scaling coefficients at coarser scales, \( j < J \), are computed recursively using a 1-d lowpass scaling filter \( \{ h(n) \} \) and a 1-d highpass filter

\[
\hat{h}(n) = (-1)^{n} h(1-n) \]

(Exploiting separability of wavelet basis functions)
\[ c_j(m',n') = \langle f, 2^j \phi(2^j x - m, 2^j y - n) \rangle \]

\[ = \sum_{m,n} h(m-2^j) h(n-2^j) c_{j+1}(m',n') \]

\[ d_j^1(m',n') = \sum_{m,n'} h_1(m-2m') h_1(n-2n') c_{j+1}(m',n') \]

\[ d_j^2(m',n') = \sum_{m',n'} h_2(m-2m') h_2(n-2n') c_{j+1}(m',n') \]

\[ d_j^3(m',n') = \sum_{m',n'} h_3(m-2m') h_3(n-2n') c_{j+1}(m',n') \]

\[ \text{Note: Decimation in both vertical and horizontal directions} \]
Organization and Display of 2d-DWT

Assume we begin with a $2^J \times 2^J$ digital image $f_J$ (i.e., \{ $c_J(m,n)$ \}_{m,n=0}^{2^J-1}$).

Because of decimation at each stage, we have

$$\{ d^j_i(m,n) \}_{m,n=0}^{2^j-1} \quad 2^j x 2^j \text{ instead of } 2^J x 2^J \quad 0 \leq j < J.$$ 

Note that the $j_0$-scale ($j_0 \leq J$) DWT of $f_J$ produces

$$\{ c_{j_0}(m,n) \}_{m,n=0}^{2^{j_0}-1} \quad \text{and} \quad \{ d^j_0(m,n) \}_{m,n=0}^{2^j-1} \quad j = j_0, \ldots, J-1$$

Exactly $2^J \times 2^J$ scaling and wavelet coefficients.
We can organize the coefficients into an image, e.g., $j_0 = 2$.

<table>
<thead>
<tr>
<th>(0,0)</th>
<th>(0, $2^{j_0}$-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ d_{j_2}^1 (m,n) }$</td>
<td>${ d_{j_2}^1 (m,n) }$</td>
</tr>
<tr>
<td>${ d_{j_2}^2 (m,n) }$</td>
<td>${ d_{j_2}^3 (m,n) }$</td>
</tr>
<tr>
<td>${ d_{j_1}^2 (m,n) }$</td>
<td>${ d_{j_1}^3 (m,n) }$</td>
</tr>
<tr>
<td>($2^{j_1}$-1, $2^{j_1}$-1)</td>
<td></td>
</tr>
</tbody>
</table>

The "subimages" are collections of wavelet or scaling coefficients at a particular scale (and orientation in the wavelet case).