Time-Frequency Analysis

Linear Analysis

Decompose the signal $x(t)$ into elementary "time-frequency atoms" (e.g., Gabor functions):

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Sx(u,f) g(t-u) e^{j2\pi ft} \, df \, du$$

"weights" Gabor functions

The weights in this superposition reflect the distribution of signal energy in time and freq.

Quadratic Analysis

Goal is to find a time-freq energy density $p_x(t, f)$ satisfying

$$\int_{-\infty}^{\infty} p_x(u, f) \, du = |X(f)|^2$$

$$\int_{-\infty}^{\infty} p_x(t, u) \, du = |x(t)|^2$$

Clearly, this requires that $p_x$ is a quadratic function of $x(t)$. 
Linear Time-frequency Analysis

The basic idea is to represent a signal \( x \) in terms of elementary "building-block" signals that are well concentrated in time and frequency. These elementary signals are called "time-frequency atoms."

Let \( \{ \phi_y \}_{y \in Y} \) be a family of TF atoms. Here, the index \( Y \) may indicate both time- and frequency localization; e.g.,

\[ Y = (t_0, f_0) \]

Let us assume that all atoms are normalized

\[ \| \phi_y \|_2 = \left( \int |\phi_y(t)|^2 dt \right)^{1/2} = 1. \]
Then we can define a linear time-frequency operator $T$ as

$$Tx(y) = \int_{-\infty}^{\infty} x(t) \overline{\Phi}_y(t) \, dt$$

$$= \langle x, \Phi_y \rangle$$

Also, by Parseval's Theorem

$$Tx(y) = \int_{-\infty}^{\infty} X(f) \overline{\Phi}_y(f) \, df$$

where $X, \Phi$ are Fourier transforms of $x, \phi$.

So, if $\Phi_y$ is concentrated in a neighborhood about time $t_0$, then $Tx(y)$ depends only on values of $x$ in this neighborhood. Similarly, if $\overline{\Phi}_y$ is concentrated in a neighborhood about frequency $f_0$, then $Tx(y)$ only depends on $X(f)$ in this neighborhood.
Ex.

1. **Windowed Fourier Transform**

   \[ \phi_x(t) = e^{j2\pi f_0 t} w(t-t_0) \]
   
   Window function

2. **Wavelet Transform**

   \[ \phi_x(t) = \frac{1}{\sqrt{s_0}} \psi \left( \frac{t-t_0}{s_0} \right) \]

3. \( \gamma = (t_0, s_0) \), \( \psi(t) = \) "mother" wavelet

   \( \text{scale instead of frequency} \)

   \[ \text{scale } s_0 < \frac{1}{f_0} \]

   smaller scale \( \leftrightarrow \) higher frequency
**Time-Frequency Concentration**

The energy of $\phi_y$ is centered at

$$t_y = \int_{-\infty}^{\infty} t |\phi_y(t)|^2 dt$$

in time, and in frequency at

$$f_y = \int_{-\infty}^{\infty} f |\overline{\Phi}_y(f)|^2 df$$

The energy "spread" about $(t_y, f_y)$ is measured by

$$\sigma_t^2(y) = \int_{-\infty}^{\infty} (t-t_y)^2 |\phi_y(t)|^2 dt$$

$$\sigma_f^2(y) = \int_{-\infty}^{\infty} (f-f_y)^2 |\overline{\Phi}_y(f)|^2 df$$

**Heisenberg Uncertainty**

$$\Rightarrow \sigma_t(y) \sigma_f(y) \geq$$
Time-Frequency Energy Density

\[ |\langle x, \phi_r \rangle |^2 = \left| \int_{-\infty}^{\infty} x(t) \phi^*_r(t) \, dt \right|^2 \]

measures the energy of \( x \) in the neighborhood of \((t_r, f_r)\).

\[ |T_x(r)|^2 \] is a time-frequency energy distribution function of \( x \).
Short-Time Fourier Transform (STFT)

Replace global Fourier analysis

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \]

with a sequence of local analyses with respect to a moving observation window

\[ S_x(t,f) = \int_{-\infty}^{\infty} x(u) e^{-j2\pi fu} w(u-t) \, du \]

window about time \( t \).

\[ = \int_{-\infty}^{\infty} X(v) W(v-f) e^{-j2\pi (v-f)t} \, dv \]

window about frequency \( f \).

Common windows

Gaussian \( \Rightarrow \) Gabor analysis
rectangle, Hamming, Blackman

\( w(t) \) assumed to be real, symmetric and \( \int |w(t)|^2 \, dt = 1 \).
\[ x(t) = e^{j2\pi f_0 t} \]

\[ \mathcal{S}_x(t, f) = \int_{-\infty}^{\infty} x(u) e^{-j2\pi fu} w(u-t) du \]

\[ = \int_{-\infty}^{\infty} x(u) W(v-\rho) e^{-j2\pi (v-f) \rho} d\rho \]

\[ x(t) = \delta(t-t_0) \]
The energy distribution resulting from the STFT is called the spectrogram

\[ P_{S_x}(t,f) = |S_x(t,f)|^2 \]

\[ = \left| \int_{-\infty}^{\infty} x(u) e^{-j2\pi fu} w(u-t) \, du \right|^2 \]

Ex. \( w(u) = \frac{1}{(\pi \sigma^2)^{1/4}} e^{-u^2/2\sigma^2}, \quad \sigma = 50 \)
Time-frequency Resolution of Spectrogram

Because \( w(u) \) is assumed to be real and symmetric, both the time and frequency spread about \((t,f)\) is independent of \((t,f)\):

\[
\sigma_t^2 = \int_{-\infty}^{\infty} (t-u)^2 |w(t-u)e^{-j2\pi fu}|^2 du
\]

\[
= \int_{-\infty}^{\infty} u^2 |w(u)|^2 du, \quad \text{indep. of } t
\]

\[
\sigma_f^2 = \int_{-\infty}^{\infty} (f-v)^2 |w(f-v)e^{-j2\pi (v-f)t}|^2 dv
\]

\[
= \int_{-\infty}^{\infty} v^2 |w(v)|^2 dv, \quad \text{indep. of } f
\]

\( \Rightarrow \) time resolution \( \sigma_t \) and frequency resolution \( \sigma_f \) are independent of position in the TF plane.