Now let's go back to a more basic question. Why frequency (spectral) analysis?

Let $x(t)$ be a deterministic, continuous-time signal. The Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{-j2\pi ft} \, df$$

shows us that this signal can be represented with an (infinite) superposition of complex sinusoids. The spectral analysis problem aimed at determining the density of this superposition from a finite number of noisy measurements of $x$. 

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But, perhaps this is the wrong question/problem in the first place. While it is true that we generally have a Fourier representation, the individual complex sinusoid components of the superposition lack a solid physical interpretation.

Complex sinusoids are:

1. everlasting
2. completely nonlocal

Most real-world signals, on the other hand, are:

1. essentially time-limited
2. localized in time
Time-frequency analysis takes a slightly different approach. It aims to describe how the frequency content (spectral density) varies in time. This raises a myriad of questions and problems — frequency means variation in time; how can frequencies vary over time?! It should sound like a slightly circular problem (because in many ways it is).

A starting point for time-frequency analysis is the following question: Can a signal be (approximately) time and band limited?
Heisenberg’s Uncertainty Principle

Does the notion of a time-frequency energy density make sense?

Such a density should measure the signal energy at different points in time and frequency.

To measure energy at

time to: \( E_{t_0} = \left( \int x(t) \delta(t-t_0) \, dt \right)^2 \)

frequency \( f_0 \): \( E_{f_0} = \left( \int x(t) e^{-j2\pi f_0 t} \, dt \right)^2 \)

The function \( \delta(t-t_0) \) is localized in time

\( e^{-j2\pi f_0 t} \) is localized in frequency
Thus, to measure the energy at time $t_0$ and frequency $f_0$, we require a function $g_{t_0,f_0}(t)$ that is localized in both time and frequency (about $(t_0,f_0)$):

$$E_{t_0,f_0} = \left( \int x(t) g_{t_0,f_0}(t) \, dt \right)^2$$

Does such a $g_{t_0,f_0}$ exist?

Let's see what we can do.
Let us postulate a function $g$ that is bandlimited to $f \in [-B/2, B/2]$ and time limited to $t \in [-T/2, T/2]$. Any non-zero function satisfying these properties would have to satisfy the relation

$$g(t) = \int_{-B/2}^{B/2} G(f) e^{-j2\pi ft} \, df = 0$$

for $|t| > T/2$.

Since $g$ is of bounded (time) duration, the same must be true for its $n^{th}$ derivative:

$$\frac{d^n g(t)}{dt^n} = \int_{-B/2}^{B/2} (j2\pi f)^n G(f) e^{j2\pi ft} \, df = 0$$

for $|t| > T/2$, $n \geq 0$.

# Bandlimited $\Rightarrow$ analytic

($l_2$, very smooth, and differentiable)
The value of \( g \) at a point \( s \in [-\frac{T}{2}, \frac{T}{2}] \) can be written as

\[
g(s) = \int_{-\frac{B}{2}}^{\frac{B}{2}} G(f) e^{j2\pi f(s-t)} e^{j2\pi ft} \, df
\]

\( |t| > \frac{T}{2} \)

By replacing \( e^{j2\pi f(s-t)} \) with its power series

\[
e^{j2\pi f(s-t)} = \sum_{n=0}^{\infty} \frac{(j2\pi(s-t))^n}{n!} f^n
\]

we have

\[
g(s) = \sum_{n=0}^{\infty} \frac{(s-t)^n}{n!} \int_{-\frac{B}{2}}^{\frac{B}{2}} (j2\pi f)^n G(f) e^{j2\pi ft} \, df
\]

\[
= \frac{d^n}{dt^n} \left( \int_{-\frac{B}{2}}^{\frac{B}{2}} G(f) e^{j2\pi ft} \, df \right) = 0
\]

since \( |t| > \frac{T}{2} \)
Thus, for all $|s| < \frac{\pi}{2}$ we have $g(s) = 0$.

This contradicts our initial assumption that the signal is non-zero in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

**Conclusion:** The only signal that is exactly time and band limited (to any intervals) is the trivial function $g \equiv 0$.

So, we must relax our strict constraint of finite (compact) support in both time and frequency.
We know that a short pulse in time extends over a broad frequency range. Vice-versa, the narrower the band of a filter, the longer its impulse response. At the extremes

\[ \delta(t) \leftrightarrow 1 \]

\[ 1 \leftrightarrow \delta(f) \]

A less extreme case is seen with Gaussian functions.

Let

\[ g(t) = \left( \frac{a}{\pi} \right)^{\frac{1}{2}} e^{-\alpha t^2} \]

Then

\[ G(f) = \int_{-\infty}^{\infty} \left( \frac{a}{\pi} \right)^{\frac{1}{2}} e^{-\alpha t^2} e^{-j2\pi ft} dt \]

\[ = e^{-\pi^2 f^2 / \alpha} \]
The "width" or essential support of $g$ in time is $\frac{1}{\sqrt{\alpha}}$ and in frequency the essential support is $\sqrt{\alpha}$.

\[ \alpha \uparrow \Rightarrow \begin{array}{c} \text{time} \\ \text{support} \end{array} \quad \begin{array}{c} \text{frequency} \\ \text{support} \end{array} \]

\[ \alpha \downarrow \Rightarrow \begin{array}{c} \text{time} \\ \text{support} \end{array} \quad \begin{array}{c} \text{frequency} \\ \text{support} \end{array} \]

What happens as $\alpha \to 0$ or $\alpha \to \infty$?
We are now in a position to pose the following question. How time and band limited can a function $g$ be? We already know that there must be a limit ($g$ can't be time and band limited).

The answer lies in Heisenberg's famous **Uncertainty Principle**.

Recall: We cannot know the position and momentum of a particle simultaneously. The more
Let us define the time and frequency concentrations of $g$ about a point $(t_0, f_0)$ as

$$\sigma^2_t = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty} (t-t_0)^2 |g(t)|^2 dt$$

$$\sigma^2_f = \frac{1}{\|g\|^2} \int_{-\infty}^{\infty} (f-f_0)^2 |G(f)|^2 df$$

**Theorem**: If $g \in L^2$ \((\|g\|^2 = \int |g(t)|^2 dt < \infty)\) then

$$\sigma^2_t \sigma^2_f \geq \frac{1}{(4\pi)^2}$$

for every $t_0, f_0$. 
Proof: We will assume that
\[ \lim_{t \to \infty} t g(t) = 0, \] implying that \( g(t) \) decays sufficiently fast.
However, the Theorem holds for any \( g \in L^2 \) (but this requires a
bit more work to prove).

First note that if \( g \) is concentrated
about \((t_0, f_0)\), then \( h(t) = e^{-j \pi f_0 t} g(t + t_0) \)
is concentrated about \((0,0)\). Therefore, without loss of generality, we can
assume that \((t_0, f_0) = (0,0)\). Let's also assume that \( \| \|_2^2 = 1 \), for convenience.

By definition
\[ \sigma_x^2 \sigma_f^2 = \int_{-\infty}^{\infty} |t g(t)|^2 dt \int_{-\infty}^{\infty} |f g(f)|^2 df \]
Next, since
\[ g'(t) \leftrightarrow j2\pi f \ G(f) \]
we can apply Parseval's Theorem to get
\[ \int_{-\infty}^{\infty} |fG(f)|^2 df = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |g'(t)|^2 dt \]
Therefore,
\[ \sigma_t^2 \sigma_f^2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |tfg(t)|^2 dt \int_{-\infty}^{\infty} |g'(t)|^2 dt \]
Now recall the Schwarz Inequality:
\[ \left( \int x(t)y^*(t) \, dt \right)^2 \leq \int |x(t)|^2 dt \cdot \int |y(t)|^2 dt \]
Applying the Schwartz inequality in our case gives

\[ \sigma_t^2 \sigma_f^2 \geq \frac{1}{4\pi^2} \left( \int_{-\infty}^{\infty} t g(t) g'(t) \, dt \right)^2 \]

Observe that

\[ t g(t) g'(t) = \frac{t}{2} \left( g^2(t) \right)' \]

Now use integration by parts

\[ \left( \int_{-\infty}^{\infty} u \, dv = uv \bigg|_{-\infty}^{\infty} - \int v \, du, \quad u = t, \quad v = \frac{g^2(t)}{2} \right) \]

\[ \int_{-\infty}^{\infty} \frac{g^2(t)}{2} \, dt = \left. t \frac{g^2(t)}{2} \right|_{-\infty}^{\infty} - \frac{1}{2} \int_{-\infty}^{\infty} g^2(t) \, dt \]

\[ = 0 \quad \text{since} \quad \lim_{t \to \infty} t g^2(t) = 0 \]

\[ \Rightarrow \]

\[ \sigma_t^2 \sigma_f^2 \geq \frac{1}{16\pi^2} \left( \int_{-\infty}^{\infty} g^2(t) \, dt \right)^2 \]

\[ = \frac{1}{(4\pi)^2} \]

\[ ||g||^2 = 1 \]
Summary: Let $x$ be a CT signal

Energy at time $t_0$:

$$E_{t_0} = \left( \int_{-\infty}^{\infty} x(t) \delta(t-t_0) \, dt \right)^2$$

$$= \langle x, \delta(\cdot-t_0) \rangle$$

Energy at frequency $f_0$:

$$E_{f_0} = \left( \int_{-\infty}^{\infty} x(t) e^{-j2\pi f_0 t} \, dt \right)^2$$

$$= \langle x, e^{-j2\pi f_0 (\cdot)} \rangle$$

Energy at time $t_0$ and frequency $f_0$:

$$E_{t_0, f_0} = \left( \int_{-\infty}^{\infty} x(t) g_{t_0,f_0}(t) \, dt \right)^2 = \langle x, g_{t_0,f_0} \rangle$$

$g_{t_0,f_0}$ cannot be totally local in both time and frequency.

Heisenberg: $\sigma_t^2 \sigma_f^2 \geq \frac{1}{(4\pi)^2}$