Non-parametric methods:

- Periodogram
- Blackman-Tukey
- Periodogram Averaging

**Key:** No assumptions on signal under study, except stationarity

Parametric methods:

* model-based *

If assumed model is a close approximation to reality, then parametric methods outperform non-parametric ones.

1. Assume signal was generated according to a parameterized model.
2. Estimate model parameters.
3. Power spectral density is then derived from estimated model.
Signals with Rational Spectra

A rational power spectral density is a ratio of two polynomials in $e^{-j2\pi f}$:

$$\Gamma(f) = \frac{\sum_{k=-q}^{q} \mu_k e^{-j2\pi fk}}{\sum_{k=-p}^{p} \xi_k e^{-j2\pi fk}}$$

where $\mu_k = \mu_k^*$ and $\xi_k = \xi_k^*$.

The Weierstrass Theorem asserts that any continuous power spectral density can be approximated arbitrarily closely by a rational function, provided the degrees $p$ and $q$ are sufficiently large.
Because $\Gamma(f) \geq 0$, the rational power spectral density can be factored as

$$\Gamma(f) = \frac{|B(f)|^2}{|A(f)|^2} \sigma^2$$

where $\sigma^2$ is a positive scalar, and $A(f)$ and $B(f)$ are the polynomials

$$A(f) = 1 + a_1 e^{-j 2\pi f} + \ldots + a_n e^{-j 2\pi f}$$

$$B(f) = 1 + b_1 e^{-j 2\pi f} + \ldots + b_m e^{-j 2\pi f}$$
Alternatively, we can express the power spectral density in the \( z \)-domain.

\[
\Gamma(z) = \frac{\sum_{k=-q}^{0} \mu_k z^{-k}}{\sum_{k=-p}^{0} \rho_k z^{-k}}
\]

or using

\[
A(z) = 1 + a_1 z^{-1} + \cdots + a_n z^{-p}
\]
\[
B(z) = 1 + b_1 z^{-1} + \cdots + b_m z^{-q}
\]

\[
\Gamma(z) = \sigma^2 \frac{B(z)B^*(\frac{1}{z^*})}{A(z)A^*(\frac{1}{z^*})}
\]

This is called the "Spectral Factorization."

Note:

\[
A^*(\frac{1}{z^*}) = \left( A(\frac{1}{z^*}) \right)^*
\]

\[
= \left( 1 + a_1 z^* + \cdots + a_n z^{*-p} \right)^*
\]

\[
= 1 + a_1^* z + \cdots + a_n^* z^p
\]
Generative Model

Recall:

\[ \{w(n)\} \rightarrow H(f) \rightarrow \{x(n)\} \]

\begin{align*}
\Gamma_{ww}(m) &= \sigma^2 \delta(m) \\
\Gamma_{ww}(f) &= \sigma^2 \\
\Gamma_{xx}(f) &= \sigma^2 \left| H(f) \right|^2
\end{align*}
In particular, if the (temporal) dynamics of the LTI system \( H \) are governed by the difference equation

\[
x(n) = -\sum_{k=1}^{p} a_k \, x(n-k) + \sum_{k=0}^{q} b_k \, w(n-k)
\]

Then

\[
H(f) = \frac{B(f)}{A(f)}
\]

and

\[
\Gamma_{xx}(f) = \sigma^2 \frac{|B(f)|^2}{|A(f)|^2}
\]

a rational power spectral density!
From
\[ \Gamma_{xx}(f) = \sigma^2 \frac{|B(f)|^2}{|A(f)|^2} \]

we see that the spectral estimation problem can be reduced to a problem of signal modeling. Moreover, the continuous function \( \Gamma_{xx} \) is actually only dependent on \( p+q+1 \) parameters
\[
\{a_1, \ldots, a_p\} \\
\{b_1, \ldots, b_q\} \\
\] and \( \sigma^2 \).

This means that every \( \Gamma_{xx} \) of the form ④ can be associated with a vector in \( (p+q+1) \) dimensional space.

Does every unique vector produce a unique \( \Gamma_{xx} \) ?
Three Signal Models:

**ARMA:**

\[ X(n) = - \sum_{k=1}^{p} a_k X(n-k) + \sum_{k=0}^{q} b_k W(n-k) \]

**AR:**

\[ X(n) = - \sum_{k=1}^{p} a_k X(n-k) + W(n) \]

**MA:**

\[ X(n) = \sum_{k=0}^{q} b_k W(n-k) \]

Spectrum estimation boils down to estimating the parameters from a finite record of data, \( \{\hat{a}_1, ..., \hat{a}_p, b_1, ..., \hat{b}_q, \sigma^2\} \) and then plugging these in to form

\[
\hat{\Gamma}(f) = \hat{\sigma}^2 \frac{|\hat{S}(f)|^2}{|\hat{A}(f)|^2}
\]
To obtain an expression for $\gamma (\tau)$ in terms of $\{a_i \gamma^2, b_k \gamma, \sigma^2\}$, first write

$$x(n) + \sum_{i=1}^{p} a_i x(n-i) = \sum_{j=0}^{q} b_j w(n-j)$$

Multiplying by $x^*(n-k)$ and taking expectations yields

$$\gamma(k) + \sum_{i=1}^{p} a_i \delta(k-i) = \sum_{j=0}^{q} b_j E[w(n-j)x^*(n-k)]$$

To evaluate $E[w(n-j)x^*(n-k)]$, note that

$$x(n) = \sum_{k=0}^{\infty} h(k) w(n-k)$$

where $\{h(k)\}$ is the impulse response of $H(f) = B(f)/A(f)$. 


Thus

\[ E\left[ w(n-j)X^*(n-k) \right] = E\left[ w(n-j) \sum_{k=0}^{\infty} h^*(k) w(n-k-l) \right] \]

\[ = \sigma^2 h^*(j-k) \]

and we have

\[ \gamma(k) + \sum_{i=1}^{p} a_i \gamma(k-i) = \sigma^2 \sum_{j=0}^{q} b_j h^*(j-k) \]

In general, \( h(k) \) is a nonlinear function of \( \{a_i\} \) and \( \{b_i\} \).

Note that if the \( \{h(k)\} \) were not present above, then the covariance is linearly related to \( \{a_i\} \) and \( \{b_i\} \).

In that case, estimates \( \{\hat{a}_i\} \) and \( \{\hat{b}_i\} \) can be found using our estimator \( \hat{\gamma}(k) \) and solving a system of linear equations.
AR Model for Spectral Estimation

If we restrict ourselves to the AR-(only) model, then the covariance equations simplify to

\[ Y(k) + \sum_{i=1}^{p} a_i Y(k-i) = \sigma^2 \mathbb{1}(k \leq 0) \]

This system of equations is linear in \( a_i \). In particular, for \( k = 0, \ldots, n \) we have the following system of equations:

\[
\begin{bmatrix}
Y(0) & Y(-1) & \cdots & Y(-n) \\
Y(1) & Y(0) & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
Y(n) & \cdots & Y(-1) & Y(0)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
Using all but the first row (above), we have the system

\[
\begin{bmatrix}
\gamma(1) \\
\vdots \\
\gamma(n)
\end{bmatrix}
+ 
\begin{bmatrix}
\gamma(0) & \cdots & \gamma(-n+1) \\
\vdots & \ddots & \vdots \\
\gamma(n-1) & \cdots & \gamma(0)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
\vdots \\
a_n
\end{bmatrix} =
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[\begin{array}{c}
\bar{c} \\
\bar{R} \\
a
\end{array}\]

The solution is

\[a = -\bar{R}^{-1}\bar{c}\]

Yule-Walker Method of Spectral Estimation

Replace \(\{\gamma(n)\}\) by \(\{\hat{\gamma}(n)\}\) (as defined on page 44) to form \(\hat{c}, \hat{R}\)

\[\Rightarrow \hat{a} = -\hat{R}^{-1}\hat{c}\]

\[\Rightarrow \hat{\Gamma}(f) = \frac{\sigma^2}{|\hat{A}(f)|^2}\]
**MA Model for Spectral Estimation**

In the MA case,

\[ \Gamma(f) = \sigma^2 |B(f)|^2. \]

Furthermore, since the data are supposed to obey the model

\[ w(n) \rightarrow B \rightarrow x(n) \]

we have

\[ \delta(k) = 0, \quad |k| > q. \]

Thus, a natural spectral estimator in this case is simply

\[ \hat{\Gamma}(f) = \sum_{k=-q}^{q} \hat{\delta}(k) e^{-j2\pi fk} \]

where, again, \( \hat{\delta}(k) \) is defined on page 49. Note that this is simply the BT estimator with a rect window of length \( 2q+1 \).
ARMA Model for Spectral Estimation

Recall that the covariance equations

$$\gamma(k) + \sum_{i=1}^{P} a_i \gamma(k-i) = \sigma^2 \sum_{j=0}^{q} b_j h^2(j-k)$$

are complicated, nonlinear functions of $a_i$ and $b_i$. Therefore, rather than attempting to directly solve for the ARMA parameters, we will take a two-step approach.

Note that since $h(n) = 0$ for $n < 0$, the covariance equations for $k > q$ simplify to

$$\gamma(k) + \sum_{i=1}^{P} a_i \gamma(k-i) = 0$$
Based on this observation, we can now state our two-step procedure.

**Step 1:**

Use equations

\[ \hat{y}(k) + \sum_{i=1}^{p} a_i \hat{y}(k-i) = 0 \quad \text{for } k > q \]

and solve Yule-Walker equations

\[ \hat{a} = - \hat{R}^{-1} \hat{r} \]

\[ \Rightarrow \quad \hat{p}_{AR}(f) = \frac{1}{|\hat{A}(f)|^2} \]

**Step 2:** Filter data \( \{x(n)\} \) by

FIR filter

\[ g(n) = \begin{cases} \hat{a}_n, & 0 \leq n \leq p \\ 0, & \text{otherwise} \end{cases} \]

Note:

\[ g(n) * x(n) = G(f) X(f) \]

\[ = \hat{A}(f) \frac{B(f)}{A(f)} W(f) \]

\[ \approx B(f) W(f) \quad \text{MA process!} \]
Using the filtered sequence
\[ y(n) = g(n) * x(n) \]
for an MA spectral estimate
\[ \hat{\Gamma}_{MA}(f) = \sum_{k=-q}^{q} \delta_{yy}(k) e^{-j2\pi fk} \]

Now combine \( \hat{\Gamma}_{AR} \) and \( \hat{\Gamma}_{MA} \) to obtain the final ARMA-based estimate
\[ \hat{\Gamma}_{ARMA}(f) = \frac{\hat{\Gamma}_{MA}(f)}{\hat{\Gamma}_{AR}(f)} \]
Ex.

\[ \chi(n) = \sum_{i=1}^{4} A_i e^{j(2\pi f_i n + \Phi_i)} + w(n) \]

\[ A_i = 1, \ i = 1, 2, 3, 4 \]

\[ \Phi_i \sim \text{Uniform}(0, 2\pi), \ \text{independent} \]

\[ w(n) \] \sim zero-mean, white noise

\[ f_1 = -0.222, \ f_2 = -0.166, \ f_3 = 0.10, \ f_4 = 0.1222 \]

\[ n = 0, \ldots, 1023 \]

**BT estimator**

\[ M = 12 \]

\[ \sigma_w^2 = 1 \]

\[ \sigma_w^2 = 0.5 \]

\[ \sigma_w^2 = 0.1 \]

**AR estimator**

\[ \sigma_w^2 = 1 \]

\[ \sigma_w^2 = 0.5 \]

\[ \sigma_w^2 = 0.1 \]