1. Verify that convolution in time is equivalent to multiplication in the z-transform domain.

2. Consider two stable and causal filters described by the difference equations

   \[ A. \quad y[n] = x[n] - \frac{1}{4}y[n-2] \]

   \[ B. \quad y[n] = x[n] - \frac{1}{2}x[n-1] + \frac{1}{6}y[n-1] + \frac{1}{3}y[n-2] \]

   a. What are the pole-zero plots for the two filters?

   b. Suppose that the two filters are placed in series, with filter B applied to the output
      of filter A. What is the pole-zero plot for the resulting overall filter.

   c. Suppose that the two filters are placed in parallel, with the overall output formed
      as the sum of the outputs of the two filters. What is the pole-zero plot for the
      resulting overall filter.

   d. Consider filter A. Does a stable and causal inverse filter exist? If so, express the
      inverse filter in difference equation form.
Consider the following nonlinear DT filter equation:

\[ y[n] = h[0]x[n] + h[1]x[n - 1] + b x[n]x[n - 1]. \]

The nonlinear filter is composed of a “two-tap” linear impulse response \( h[0], h[1] \) and a “bilinear” term \( b x[n]x[n - 1] \), where \( b \) is an unknown parameter reflecting the strength of the nonlinear interaction between \( x[n] \) and \( x[n - 1] \). Assuming that \( x[n] \) is still a Gaussian white noise sequence and ignoring quantization errors, can you formulate a statistical correlation method that will allow us to identify \( h[0], h[1], \) and \( b \)?
4. Image Deconvolution and Quantization Effects

Recall the following image deconvolution problem that we examined in the first exam. A special microscope is used to image viral molecules suspended in a solution. The ideal image (with five molecules) is depicted below. Unfortunately, the microscope cannot be perfectly focused and, instead of the ideal image, we measure a blurred image (also shown below).

![Ideal Image](image1.png)  ![Blurred Image](image2.png)

Recall that by using the known or estimated PSF of the imaging system, we can restore the image as follows. Let $h[m, n]$ denote the 2-d PSF of the imaging system, and let $H[k, \ell]$ denote its 2-d DFT. Also let $Y[k, \ell]$ denote the 2-d DFT of the blurred image. Then the original image can be approximately recovered (perfectly except for minor circular convolution effects) by forming the ratio $X[k, \ell] = \frac{Y[k, \ell]}{H[k, \ell]}$ and taking the inverse 2-d DFT.

We know from earlier homework that this sort of deconvolution process can breakdown if noise is added to the blurry image. The problem we will consider here is what happens if the blurry image is quantized prior to deconvolution.

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To investigate the effects of quantization, let us make the following assumptions. The 2D Fourier transforms of the bead and blurred bead images are radially symmetric (because the bead and the PSF are known to be radially symmetric); that is, the 2-d DTFT of a single bead, denoted $B(u, v)$, is just a function of $r = \sqrt{u^2 + v^2}$ and not angle. The radial profile of $|B(u, v)|^2$ and the radial profile of the squared magnitude of the DTFT of blurred bead are shown below (from $r = 0$ to $r = \pi$). The pixel gray values of the images range between 0 and 100. Suppose that after acquisition the image was stored on a computer for further processing. Each pixel is quantized with 8 bit accuracy (assume a uniform quantization step size).

What is the quantization noise power $\sigma^2_e$? On the plot of the DTFT magnitudes below, sketch the radial power spectrum for quantization noise.

![DTFT profile of bead and blurred bead](image)

The presence of the quantization noise makes determination of the PSF much more challenging. Propose a method for estimating the PSF from the quantized images that accounts for the presence of quantization noise.

Propose a frequency domain filtering scheme that can be applied to the quantized blurry image. The filter should aim to strike a balance between “resolution recovery” (i.e., deblurring) and high-frequency noise amplification.
5. Consider the system

\[ y[n] = x[n] - ax[n - 1] \]

where \( 0 < a < 1 \). Suppose that the input to the system, \( x[n] \), is a wide-sense stationary process with \( E[x[n]] = 0 \) and \( R_{xx}[m] = E[x[n]x[n - m]] = \frac{a^{|m|}}{1 - a^2} \). Find the mean and autocorrelation function for the output \( y[n] \).

6. System identification. Suppose that a Gaussian white noise signal with unit power, \( w[n] \), is the input to an LTI discrete-time filter, and denote the output by \( x[n] \).

a. Show that the filter’s frequency response can be easily recovered from the Fourier transform of the cross-correlation function \( R_{wx}[m] = E[w[n]x[n + m]] \).

b. Suppose that the output’s mean and autocorrelation function are \( E[x[n]] = 0 \) and \( R_{wn}[m] = E[w[n]x[n + m]] = a^m u[n] \), \( |a| < 1 \). What is the filter in this case?